Flavor ratios of astrophysical neutrinos: implications for precision measurements

Sandip Pakvasa

Department of Physics and Astronomy, University of Hawaii,
Honolulu, HI 96822, U.S.A.
E-mail: `pakvasa@phys.hawaii.edu`

Werner Rodejohann

Max-Planck-Institut für Kernphysik,
Postfach 103980, D-69029 Heidelberg, Germany
E-mail: `werner.rodejohann@mpi-hd.mpg.de`

Thomas J. Weiler

Department of Physics and Astronomy, Vanderbilt University,
Nashville, TN 37235, U.S.A.
E-mail: `tom.weiler@vanderbilt.edu`

ABSTRACT: We discuss flavor-mixing probabilities and flavor ratios of high energy astrophysical neutrinos. In the first part of this paper, we expand the neutrino flavor-fluxes in terms of the small parameters U_{e3} and $\pi/4 - \theta_{23}$, and show that there are universal first and second order corrections. The second order term can exceed the first order term, and so should be included in any analytic study. We also investigate the probabilities and ratios after a further expansion around the tribimaximal value of $\sin^2 \theta_{12} = 1/3$. In the second part of the paper, we discuss implications of deviations of initial flavor ratios from the usually assumed, idealized flavor compositions for pion, muon-damped, and neutron beam sources, viz., $(\nu_e : \nu_\mu : \nu_\tau) = (1 : 2 : 0)$, $(0 : 1 : 0)$, and $(1 : 0 : 0)$, respectively. We show that even small deviations have significant consequences for the observed flavor ratios at Earth. If initial flavor deviations are not taken into account in analyses, then false inferences for the values in the PMNS matrix elements (angles and phase) may result.

KEYWORDS: High Energy Cosmic Rays, Neutrino Physics.

Contents

1. Introduction

There has been recently much discussion on the flavor mixing of high energy astrophysical neutrinos [1–15]. Neutrino mixing modifies the initial flavor distribution of fluxes Φ_e^0 : Φ_μ^0 : Φ_τ^0 in calculable ways. In terrestrial neutrino telescopes such as IceCube [16] or KM3Net [17], one can measure neutrino flavor ratios [18] and thereby obtain information on the neutrino parameters and/or the sources.[1] There are two essential ingredients to such analyses. One is the initial flavor composition which depends on the nature of the production process at the source. The other is the neutrino mixing scheme, in particular the values of the parameters governing neutrino mixing. The latter plays the role of altering the flavor mix from the original due to non-trivial lepton mixing. We will discuss in this paper precision issues for both these aspects.

---

[1]In principle, more exotic neutrino properties such as neutrino decay, Pseudo-Dirac structure, magnetic moments, interaction with dark energy, or breakdown of fundamental symmetries could also be studied [19–24].

The current best-fit values as well as the allowed 1σ and 3σ ranges of the oscillation parameters are [25]:

$$\sin^2\theta_{12} = 0.32^{+0.02, 0.08}_{-0.02, 0.06},$$
$$\sin^2\theta_{23} = 0.45^{+0.07, 0.20}_{-0.07, 0.13},$$
$$|U_{e3}|^2 < 0.02 \ (0.04) \ . \tag{1.1}$$

The CP phase is completely unknown and has a range between zero and 2π. The current information on the mixing parameters [26] therefore suggests that θ_{13} and the deviation from maximal atmospheric neutrino mixing are small parameters and therefore to expand the formulae in terms of them. This is quite useful in order to obtain an analytical understanding of basically all phenomenological problems of interest. Furthermore, the deviation from the value $\sin^2\theta_{12} = \frac{1}{3}$ is also small and thereby a third small expansion parameter is introduced. An idealized description for the leptonic mixing, or Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix U, which is nevertheless perfectly compatible with all experimental information is tribimaximal mixing [27]

$$U \simeq U_{\mathrm{TBM}} = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 & \sqrt{2} & 0 \\ -1 & \sqrt{2} & -\sqrt{3} \\ -1 & \sqrt{2} & \sqrt{3} \end{pmatrix} \ . \tag{1.2}$$

The transition probabilities of astrophysical neutrinos are functions of the elements of U, whose usual PDG parameterization is given in the appendix. Expanding these probabilities in terms of the small parameters motivated by the observed oscillation phenomenology is one of the purposes of this paper. Previously, expansions in terms of $|U_{e3}|$ and $\pi/4 - \theta_{23}$ have been discussed. In this case, we will show here first that in addition to the known universal first order correction [8, 11] there is a second order universal correction which can exceed the first order one and has to be included in analytical studies. Furthermore, by expanding the probabilities in terms of the deviation from $\sin^2\theta_{12} = \frac{1}{3}$ we find extremely compact expressions.

In what concerns the initial flavor mix at the source, there are three simple and — as we will make clear in this paper — idealized possibilities:

- The most conventional one is neutrino emission from purely hadronic processes, such as $p + p \to \pi^{\mp} \to \mu^{\mp} + \overset{(-)}{\nu_\mu} \to e^{\mp} + \overset{(-)}{\nu_e} + \nu_\mu + \overline{\nu}_\mu$, in which the initial neutrino flavor mix is identical to the one in atmospheric neutrinos:

$$\Phi^0_e : \Phi^0_\mu : \Phi^0_\tau = 1 : 2 : 0 \ . \tag{1.3}$$

  If the dominant decay process is $p + \gamma \to \pi^+ + X$ then the flavor mix is still $1 : 2 : 0$, however the initial $\overline{\nu}_e$ flux is absent. This can in principle be checked by taking advantage of the Glashow resonance reaction $\overline{\nu}_e + e^- \to W^-$ [1, 28, 3], which occurs at an incident $\overline{\nu}_e$ energy of 6.3 PeV.

- The muons may lose energy so that the $\overset{(-)}{\nu_e}$ flux is depleted at the energies of interest. This can happen in a variety of ways [29 – 34, 14]; e.g., muons can lose energy in

strong magnetic fields, or get absorbed in matter. In these cases the initial flavor composition is simply

$$\Phi_e^0 : \Phi_\mu^0 : \Phi_\tau^0 = 0 : 1 : 0 \qquad (1.4)$$

without any electron or tau neutrinos. Specific models have been discussed. In general, one expects that any pion source will have a transition to a muon-damped composition, with the transition energy depending on the source properties. Qualitatively, muon damping occurs when the muon's energy-loss length $\sim \frac{E}{dE/dx}$ is shorter than its decay length $(E/m_\mu)\, c\tau_0$, i.e., when $dE/dx \gtrsim c\tau_0/m_\mu$.

- The third case is that of sources which emit dominantly neutrons. These neutrons originate in the photo-dissociation of heavy nuclei. The decays of the neutrons give rise to an initial pure "β-beam" of $\overline{\nu}_e$ [35], i.e.,

$$\Phi_e^0 : \Phi_\mu^0 : \Phi_\tau^0 = 1 : 0 : 0 \,, \qquad (1.5)$$

with no ν_e or any muon or tau neutrinos.

One other expected source of both pion-decay neutrinos and neutron-decay neutrinos, in separated energy regions [36], is the GZK nucleonic reaction $p_{\rm CR} + \gamma_{\rm CMB} \to \Delta^+ \to n + \pi^+$. The subsequent pion and muon decays produce neutrinos with energies about 20 times (i.e., $m_\pi/m_N \times 1/4$) below $E_{\rm GZK} \sim 5 \times 10^{19}$ eV, while neutron decay produces $\overline{\nu}_e$'s with energies about a thousand times (i.e., β-decay Q-value/m_n) below $E_{\rm GZK}$.

We will emphasize in the present work that the above three flavor mixes in eqs. (1.3), (1.4), (1.5) are idealizations. Realistically, one should expect deviations from these simple flux compositions. These deviations should be taken into account in analyses to avoid incorrect conclusions about the violation/conservation of CP, the octant of θ_{23}, or the magnitude of $|U_{e3}|$. We shall outline the need for care with several examples.

This paper is built up as follows. In section 2 we discuss an expansion of the mixing probabilities and flux ratios up to second order (i) in the small parameters $\sin^2 \theta_{23} - \frac{1}{2}$ and $|U_{e3}|$, and eventually, (ii) in the small parameter $\sin^2 \theta_{12} - \frac{1}{3}$. In section 3 we assess the validity of the idealized and often used initial flavor compositions given in eqs. (1.3), (1.4), (1.5). We illustrate the effects of impure neutrino mixes with various examples. We sum up and conclude in section 4.

2. Mixing probabilities and flux ratios up to second order

We discuss in this section approximate formulae for the mixing probabilities and flux ratios, and general properties of the relevant flux ratios. For illustration and insight, we will expand formulas in terms of small parameters, but we use the exact expressions for all plots of flux ratios presented in this paper.

An expansion up to second order in the small parameters[2] reveals universal correction terms [37]. We will expand first in terms of the two parameters $\sin^2 \theta_{23} - \frac{1}{2}$ and $|U_{e3}|$, related

---

[2]The importance of second order terms has also been mentioned in ref. [12].

to breaking of the $\nu_\mu \leftrightarrow \nu_\tau$ symmetry [38]. To reveal the dependence on these two parameters, it is sometimes useful to fix the solar neutrino mixing with the phenomenologically valid relation $\sin^2 \theta_{12} = \frac{1}{3}$.

2.1 Mixing probabilities

2.1.1 Expansion in terms of $|U_{e3}|$ and $\epsilon \equiv \frac{\pi}{4} - \theta_{23}$

The distances to most high energy neutrino sources are quite large compared to oscillation lengths $\lambda_{jk} = 4\pi \, E/\Delta m_{jk}^2$, where $\Delta m_{jk}^2 = m_j^2 - m_k^2$ is the mass-squared difference and E the neutrino energy. Even for energies as high as the GZK-cutoff $\sim 5 \times 10^{19}\,\mathrm{eV}$, this is so. Consequently, the terms involving the mass-squared differences in the oscillation probabilities are effectively averaged out; oscillation probabilities are reduced to mixing probabilities. The $\nu_\alpha \leftrightarrow \nu_\beta$ mixing probabilities are

$$P_{\alpha\beta} = \sum_i |U_{\alpha i}|^2 \, |U_{\beta i}|^2 \; . \tag{2.1}$$

Starting from an initial flux composition $\Phi_e^0 : \Phi_\mu^0 : \Phi_\tau^0$, the measurable neutrino flux at Earth is given by

$$\Phi_\alpha = \sum_\beta P_{\alpha\beta} \, \Phi_\beta^0 \,. \tag{2.2}$$

If tribimaximal mixing (given in eq. (1.2)) is assumed, then the flavor-propagation probabilities are simply

$$P_{\mathrm{TBM}} = \frac{1}{18} \begin{pmatrix} 10 & 4 & 4 \\ 4 & 7 & 7 \\ 4 & 7 & 7 \end{pmatrix} \; . \tag{2.3}$$

The flavor ordering for both column and row indices of this symmetric matrix are $(e, \, \mu, \, \tau)$.

Neglecting the small parameters $\pi/4 - \theta_{23}$ and θ_{13}, the probabilities $P_{\alpha\beta}$ (and therefore the observable flux ratios) are functions solely of the solar neutrino mixing angle. The exact expressions for the mixing probabilities (given in the appendix) are fourth order polynomials in the $\sin\theta_{ij}$. It is therefore more useful to expand the formulae in terms of small parameters and truncate after the quadratic terms. We will first expand in terms of

$$|U_{e3}| \;\; \text{and} \;\; \epsilon \equiv \frac{\pi}{4} - \theta_{23} = \frac{1}{2} - \sin^2\theta_{23} + \mathcal{O}(\epsilon^3) \; . \tag{2.4}$$

The explicit PDG parametrization of the PMNS matrix is given in the appendix. Writing the result in matrix form yields:

$$
P \equiv \begin{pmatrix} P_{ee} & P_{e\mu} & P_{e\tau} \\ P_{e\mu} & P_{\mu\mu} & P_{\mu\tau} \\ P_{e\tau} & P_{\mu\tau} & P_{\tau\tau} \end{pmatrix} \simeq \begin{pmatrix} 1 - 2\,c_{12}^2\,s_{12}^2 & c_{12}^2\,s_{12}^2 & c_{12}^2\,s_{12}^2 \\ c_{12}^2\,s_{12}^2 & \frac{1}{2}\left(1 - c_{12}^2\,s_{12}^2\right) & \frac{1}{2}\left(1 - c_{12}^2\,s_{12}^2\right) \\ c_{12}^2\,s_{12}^2 & \frac{1}{2}\left(1 - c_{12}^2\,s_{12}^2\right) & \frac{1}{2}\left(1 - c_{12}^2\,s_{12}^2\right) \end{pmatrix} \tag{2.5}
$$
$$
- \frac{1}{2}\left(1 - 2\,c_{12}^2\,s_{12}^2\right)|U_{e3}|^2 \begin{pmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix} + \Delta \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} + \frac{1}{2}\overline{\Delta}^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} ,
$$

As it must, every row and every column sums to 1. The correction linear in $|U_{e3}|$ and ϵ is given by [8, 11]

$$\Delta \equiv \frac{1}{4} \sin 4\theta_{12} \cos \delta \, |U_{e3}| + \frac{1}{2} \sin^2 2\theta_{12} \, \epsilon \,, \tag{2.6}$$

and the correction quadratic in $|U_{e3}|$ and ϵ is

$$\overline{\Delta}^2 \equiv \sin^2 2\theta_{12} \cos^2 \delta \, |U_{e3}|^2 + 4 \left(1 - \cos^2 \theta_{12} \sin^2 \theta_{12}\right) \epsilon^2 - \sin 4\theta_{12} \cos \delta \, |U_{e3}| \, \epsilon \quad (2.7)$$
$$= \left(\sin 2\theta_{12} \cos \delta \, |U_{e3}| - \epsilon \cos 2\theta_{12}\right)^2 + 3 \, \epsilon^2 \,.$$

The latter is a new result, as is the observation that the universal second order correction $\overline{\Delta}^2$ is positive semidefinite. With the current 1σ and (3σ) ranges of the oscillation parameters, one finds the following ranges for Δ and $\overline{\Delta}^2$:

$$(-0.104) - 0.043 \le \Delta \le 0.069 \, (0.117), \quad \text{and} \quad \overline{\Delta}^2 \le 0.061 \, (0.179) \,. \tag{2.8}$$

It is seen that the second order correction can exceed the first order correction. Consequently, the first order correction to the flux ratios alone is not sufficient to accurately describe the phenomenology. The second order correction needs inclusion in analytical studies. The reason for the large second order term are the sizable numerical coefficients, especially the one in front of ϵ^2 (we have checked that the higher order terms have smaller coefficients).

Explicitly, the individual mixing probabilities are

$$\begin{aligned}
P_{ee} &\simeq (1 - 2 \, c_{12}^2 \, s_{12}^2)(1 - 2 \, |U_{e3}|^2) \,, \\
P_{e\mu} &\simeq c_{12}^2 \, s_{12}^2 + \Delta + (1 - 2 \, c_{12}^2 \, s_{12}^2) \, |U_{e3}|^2 \,, \\
P_{e\tau} &\simeq c_{12}^2 \, s_{12}^2 - \Delta + (1 - 2 \, c_{12}^2 \, s_{12}^2) \, |U_{e3}|^2 \,, \\
P_{\mu\mu} &\simeq \frac{1}{2} \left(1 - c_{12}^2 \, s_{12}^2\right) - \Delta + \frac{1}{2} \overline{\Delta}^2 - \frac{1}{2} (1 - 2 \, c_{12}^2 \, s_{12}^2) \, |U_{e3}|^2 \,, \\
P_{\mu\tau} &\simeq \frac{1}{2} \left(1 - c_{12}^2 \, s_{12}^2\right) - \frac{1}{2} \overline{\Delta}^2 - \frac{1}{2} (1 - 2 \, c_{12}^2 \, s_{12}^2) \, |U_{e3}|^2 \,, \\
P_{\tau\tau} &\simeq \frac{1}{2} \left(1 - c_{12}^2 \, s_{12}^2\right) + \Delta + \frac{1}{2} \overline{\Delta}^2 - \frac{1}{2} (1 - 2 \, c_{12}^2 \, s_{12}^2) \, |U_{e3}|^2 \,.
\end{aligned} \tag{2.9}$$

This may be compared with the same in the case of exact tribimaximal mixing, given in eq. (2.3). The explicit terms depending on $|U_{e3}|^2$ are small, because the maximal value of $(1 - 2 \, c_{12}^2 \, s_{12}^2) |U_{e3}|^2$ is 0.011 (0.031) at 1σ (3σ), which is well below the maximal values of $|\Delta|$ and $\overline{\Delta}^2$. In addition, for the initial flavor composition of $\Phi_e^0 : \Phi_\mu^0 : \Phi_\tau^0 = 1 : 2 : 0$, the terms with explicit $|U_{e3}|^2$ cancel in the fluxes. Consequently, the first and second order corrections Δ and $\overline{\Delta}^2$ attain a universal status. P_{ee} and $P_{\mu\tau}$ receive only quadratic corrections, where the correction to P_{ee} depends only[3] on θ_{13} and not on δ or θ_{23}. Note that the universal second order term shows up only in $P_{\mu\mu}$, $P_{\mu\tau}$ and $P_{\tau\tau}$, i.e., in the μ–τ sector. This will be of interest later when we discuss neutron beam sources, in which these probabilities, and therefore $\overline{\Delta}^2$, do not show up in the flux ratios.

---

[3]This remains true for P_{ee} even when oscillations do not average out.

Inserting the tribimaximal value $\sin^2\theta_{12} = \frac{1}{3}$ in the expressions for Δ and $\overline{\Delta}^2$ gives a more illustrative result. We define the resulting universal corrections as

$$\Delta_{\text{TBM}} = \frac{1}{9}\left(\sqrt{2}\,\cos\delta\,|U_{e3}| + 4\,\epsilon\right), \tag{2.10}$$

$$\overline{\Delta}^2_{\text{TBM}} = \frac{4}{9}\left(2\cos^2\delta\,|U_{e3}|^2 + 7\,\epsilon^2 - \sqrt{2}\,\cos\delta\,|U_{e3}|\,\epsilon\right).$$

If for instance $\theta_{13} = 0$ and $|\epsilon|$ is larger than $1/7$, then $\overline{\Delta}^2_{\text{TBM}}$ exceeds $|\Delta_{\text{TBM}}|$. Note that $\Delta = \Delta_{\text{TBM}}$ plus quadratic terms and that $\overline{\Delta}^2 = \overline{\Delta}^2_{\text{TBM}}$ plus cubic terms. Hence, the ranges of Δ_{TBM} and $\overline{\Delta}^2_{\text{TBM}}$ are changed little from the ranges of Δ and $\overline{\Delta}^2$ given in eqs. (2.8).

It is also interesting to consider the allowed range of the second order correction $\overline{\Delta}^2$ in the case of a vanishing first order correction Δ. We find that, at 1σ and (3σ),

$$\text{if } \Delta = 0, \text{ then } \overline{\Delta}^2 \leq 0.029\ (0.093). \tag{2.11}$$

In particular if the oscillation parameters lie outside their current 1σ ranges, then the mixing probabilities and flux ratios can deviate up to ten percent from their tribimaximal values even if the first order correction vanishes.

Note that in the definition of Δ in eq. (2.6), the factor in front of $\frac{1}{2} - \sin^2\theta_{23}$ is larger and has a smaller range than the factor in front of $|U_{e3}|\cos\delta$. To be precise, for the allowed 3σ range of solar neutrino mixing, $\frac{1}{4}\sin 4\theta_{12}$ ranges from 0.12 to 0.21, whereas $\frac{1}{2}\sin^2 2\theta_{12}$ ranges from 0.38 to 0.48. Consequently, the sensitivity to deviations from maximal atmospheric neutrino mixing is better than the sensitivity to deviations from $|U_{e3}| = 0$. The latter is also mitigated by the dependence on the unknown CP phase δ. Comparing $\overline{\Delta}^2$ from eq. (2.7) with the first order parameter Δ, we note that the factor in front of ϵ^2 in $\overline{\Delta}^2$ is larger than the one in front of ϵ in Δ. Also in $\overline{\Delta}^2$, the factor in front of ϵ^2 has a smaller range than the one in front of $|U_{e3}|^2$: $\sin^2 2\theta_{12}$ lies between 0.75 and 0.64, whereas $4\,(1 - \cos^2\theta_{12}\sin^2\theta_{12})$ ranges from 3.06 to 3.25, when the allowed 3σ values of θ_{12} are inserted. Again, sensitivity to deviations from maximal atmospheric neutrino mixing is favored over sensitivity to $|U_{e3}|$. In figure 1 we show the minimal and maximal allowed values of Δ and $\overline{\Delta}^2$ as a function of the neutrino mixing parameters. There is almost no dependence on θ_{12}. The strongest dependence is on θ_{23}. The statistically weak preference for $\sin^2\theta_{23} = 0.45$ would mean for $\theta_{13} = 0$ that $\Delta \simeq 0.02$ is positive and $\overline{\Delta}^2 \simeq 0.008$. We show in figure 2 the distribution of $|U_{e3}|\cos\delta$ against $\sin^2\theta_{23}$ for several values of Δ, where we allow all mixing angles to vary in their allowed 3σ ranges. We also indicate the 1σ ranges for $\sin^2\theta_{23}$ and for $|U_{e3}|\cos\delta$ if $\delta = 0, \pi/4$ and $\pi/3$. Figure 3 shows the same for $\overline{\Delta}^2$. Only if $\epsilon = |U_{e3}|\cos\delta = 0$ can $\overline{\Delta}^2$ vanish exactly. The first order correction Δ may also vanish, if [8, 11]

$$|U_{e3}|\,\cos\delta = \left(\sin^2\theta_{23} - \frac{1}{2}\right)\tan 2\theta_{12}. \tag{2.12}$$

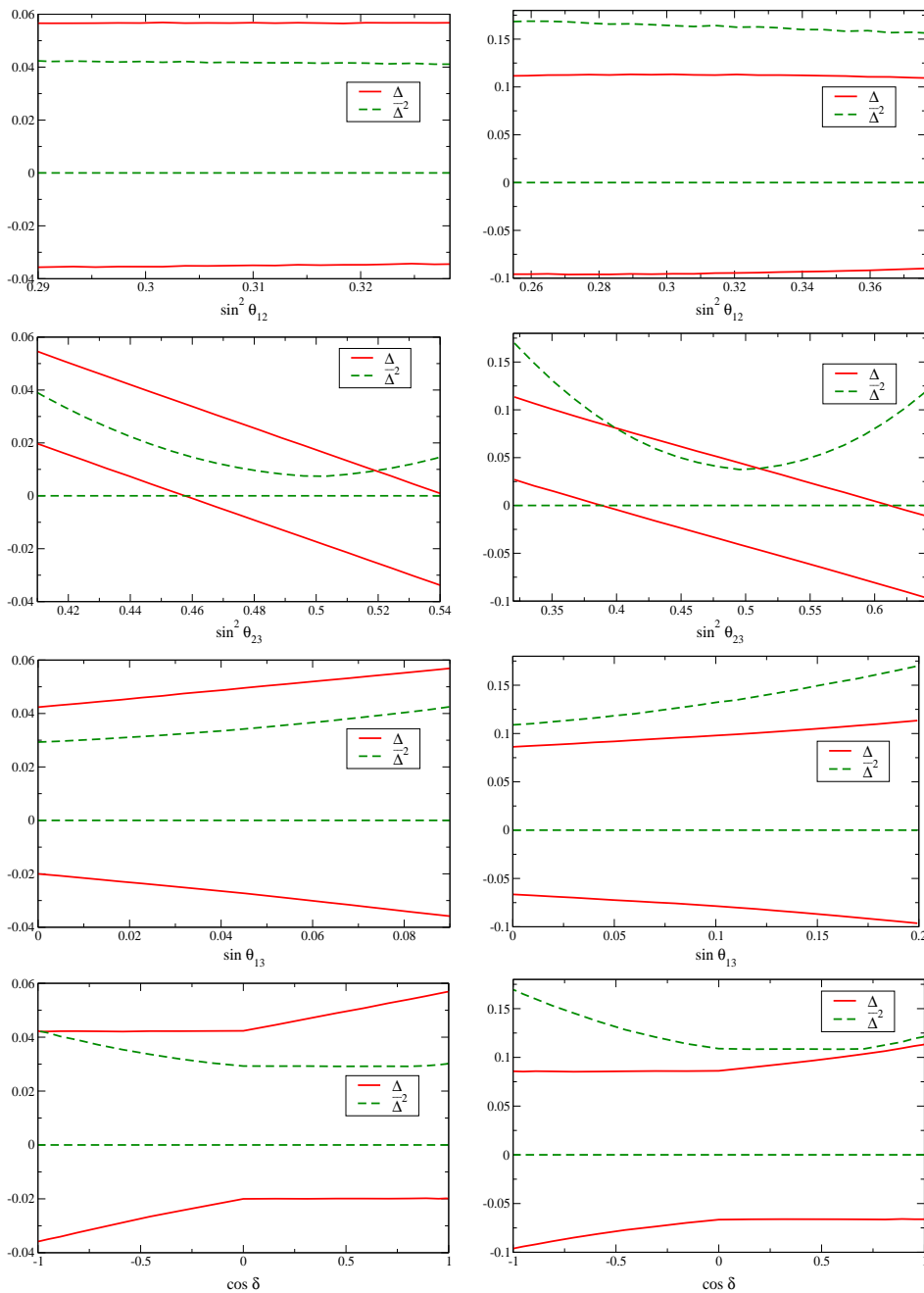The tribimaximal value of $\tan 2\theta_{12}$ is $2\sqrt{2}$.

Figure 1: The minimal and maximal allowed values of the universal first and second order parameters Δ and $\overline{\Delta}^2$ as a function of the neutrino mixing parameters. The observables not specified in the horizontal axis were varied over their currently allowed 1σ (top) and 3σ (bottom) ranges.

2.1.2 Expansion in terms of $|U_{e3}|$, $\epsilon \equiv \frac{\pi}{4} - \theta_{23}$, and $\epsilon' \equiv \sin^2 \theta_{12} - \frac{1}{3}$

We can simplify the expressions even more when we introduce a third small parameter,
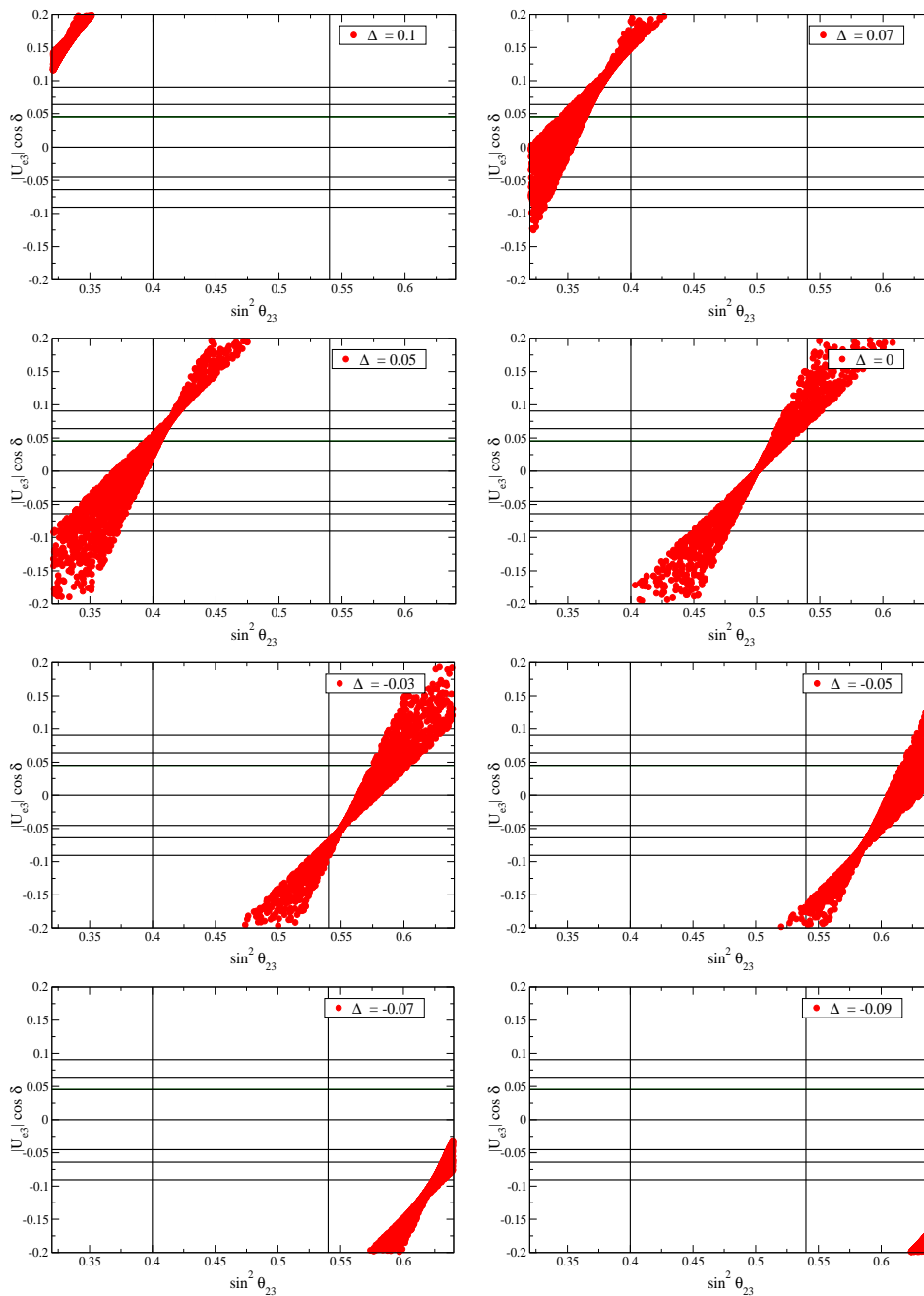
Figure 2: Distribution of $|U_{e3}| \cos \delta$ against $\sin^2 \theta_{23}$ if Δ takes certain indicated values. Indicated also is the allowed 1σ range of θ_{23} and of $|U_{e3}| \cos \delta$ for (from top to bottom above zero) $\delta = 0$, $\delta = \pi/4$ and $\delta = \pi/3$. The value $\delta = \pi/2$ means $|U_{e3}| \cos \delta = 0$.

taking advantage of the closeness of $\sin^2 \theta_{12}$ to $\frac{1}{3}$. Let us define[4]

$$\epsilon' \equiv \arcsin \sqrt{\frac{1}{3}} - \theta_{12} = \frac{3}{2\sqrt{2}} \left(\frac{1}{3} - \sin^2 \theta_{12} \right) + \mathcal{O}(\epsilon'^2) . \tag{2.13}$$

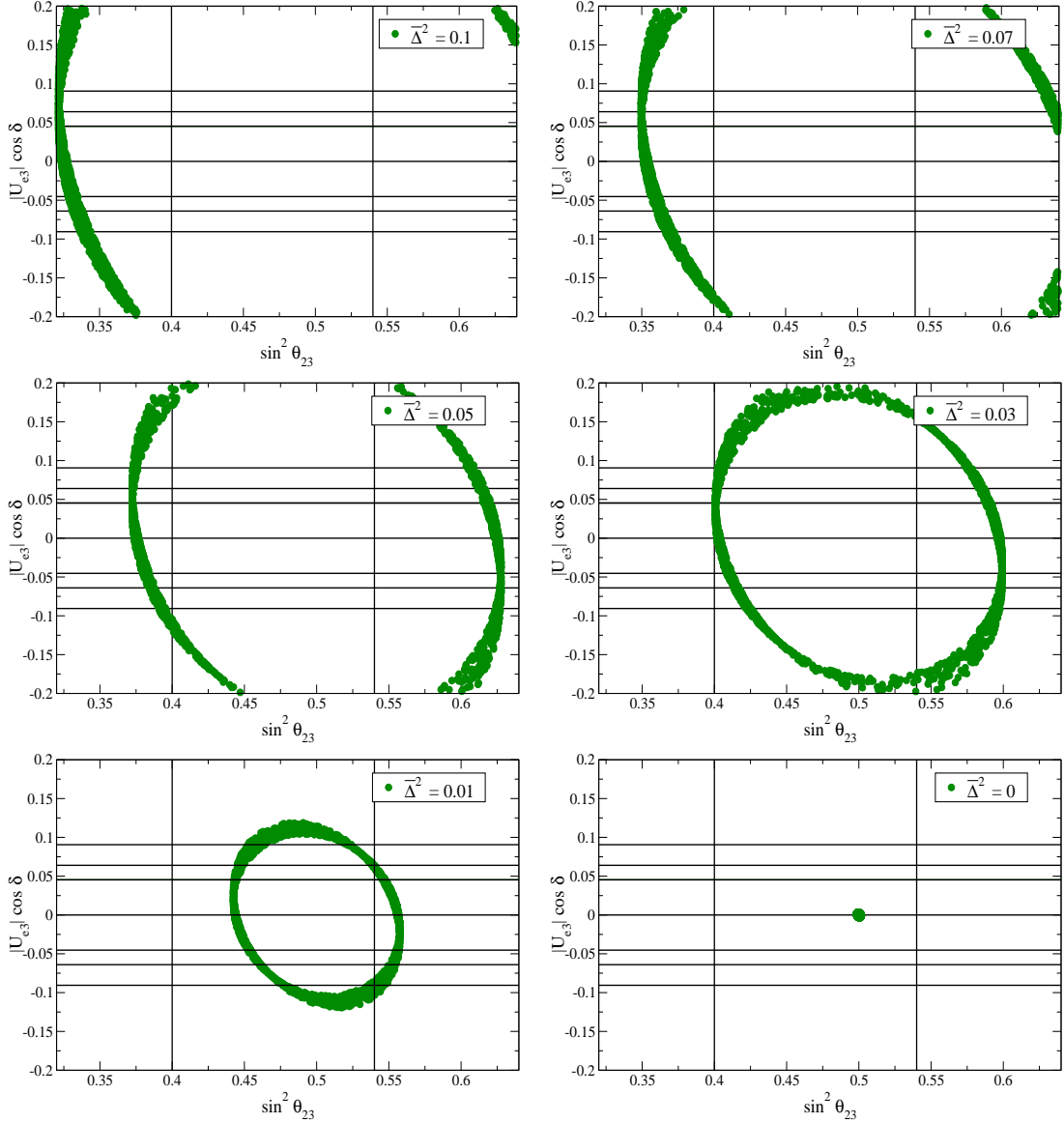[4]A related expansion can be found in ref. [37].

Figure 3: Distribution of $|U_{e3}| \cos \delta$ against $\sin^2 \theta_{23}$ if $\overline{\Delta}^2$ takes certain indicated values. Indicated also is the allowed 1σ range of θ_{23} and of $|U_{e3}| \cos \delta$ for (from top to bottom above zero) $\delta = 0$, $\delta = \pi/4$ and $\delta = \pi/3$. The value $\delta = \pi/2$ means $|U_{e3}| \cos \delta = 0$.

The best-fit value of $\sin^2 \theta_{12} = 0.32$ corresponds to $\epsilon' = 0.0142$. From the previous subsection it is not difficult to obtain the elements of the flavor-propagation matrix P. They are [37]

$$P_{ee} = \frac{1}{18} \left(10 + 4\,A \right), \qquad P_{e\mu} = \frac{1}{18} \left(4 - 2\,A + B \right), \quad P_{e\tau} = \frac{1}{18} \left(4 - 2\,A - B \right), \tag{2.14}$$

$$P_{\mu\mu} = \frac{1}{18} \left(7 + A - B + C \right), \quad P_{\mu\tau} = \frac{1}{18} \left(7 + A - C \right), \qquad P_{\tau\tau} = \frac{1}{18} \left(7 + A + B + C \right),$$

where

$$A = 2\sqrt{2}\,\epsilon' + 7\,\epsilon'^2 - 5\,|U_{e3}|^2\,,$$
$$B = 2\left(\sqrt{2}\,|U_{e3}|\cos\delta + 4\epsilon - 4\sqrt{2}\,\epsilon\,\epsilon' + 7\,\epsilon'\,|U_{e3}|\cos\delta\right)\,, \qquad (2.15)$$
$$C = 4\left(2\,|U_{e3}|^2\cos^2\delta + 7\,\epsilon^2 - \sqrt{2}\,\epsilon\,|U_{e3}|\cos\delta\right)\,.$$

Note that $C = 9\overline{\Delta}^2_{\mathrm{TBM}}$, the latter being previously defined in eq. (2.10). If θ_{23} is maximal and $|U_{e3}|\cos\delta$ vanishes, then $B = C = 0$ and only $A \neq 0$, unless in addition $\sin^2\theta_{12} = \frac{1}{3}$. Just as with $\overline{\Delta}^2$, the correction factor C is positive semi-definite and appears exclusively in the μ–τ sector. The probability P_{ee} receives contributions only from A, whereas $P_{\mu\tau}$ is not corrected by B. Deviations from $\sin^2\theta_{12} = \frac{1}{3}$ show up mainly in A.

At 1σ and (3σ), we obtain the ranges of these (A, B, C) parameters. The ranges are

$$(-0.418) \; -0.126 \leq A \leq 0.117 \; (0.284)\,, \qquad (2.16)$$
$$(-1.956) \; -0.778 \leq B \leq 1.255 \; (2.177)\,,$$
$$0 \leq C \leq 0.465 \; (1.356)\,.$$

The tribimaximal values for these parameters are zero.

Both expansions, one in terms of $(\Delta, \overline{\Delta}^2)$ and the other in terms of (A, B, C), are useful. To disentangle the dependence on θ_{12} from the dependences on $|U_{e3}|$ and $\theta_{23} - \pi/4$, the expansion defined by $(\Delta, \overline{\Delta}^2)$ may be more helpful; the $(\Delta, \overline{\Delta}^2)$ expansion has the full dependence on θ_{12} included in it. On the other hand, the range of ϵ' is smaller than the range of ϵ and $|U_{e3}|$. Therefore, to a good approximation one may set $\sin^2\theta_{12}$ to $\frac{1}{3}$. Below, we concentrate on the dependences on $|U_{e3}|$ and $\epsilon \equiv \pi/4 - \theta_{23}$.

2.2 Flavor ratios

With the help of the probabilities in eq. (2.9) and a given initial flavor composition $\Phi_e^0 : \Phi_\mu^0 : \Phi_\tau^0$ it is easy to obtain approximate formulae for flux compositions or ratios. We will focus here on the ratio of muon neutrinos to the total flux Φ_{tot}, and on the ratio of ν_e to ν_τ [18, 39]:

$$T \equiv \frac{\Phi_\mu}{\Phi_{\mathrm{tot}}} \quad\text{and}\quad R \equiv \frac{\Phi_e}{\Phi_\tau}\,. \qquad (2.17)$$

Muon neutrinos with energies $\gtrsim 10^2\,\mathrm{GeV}$ can be identified via muons emerging from the shower. Electromagnetic showers from ν_e charged current reactions may be identifiable. At energies $\gtrsim 10^6\,\mathrm{GeV}$, ν_τ can be identified by double-bang or lollipop signatures. Sometimes the ratio $\Phi_\mu/(\Phi_e + \Phi_\tau)$ is discussed in the literature. It is trivially related to our ratio T, being equal to $T/(1 - T)$.

In the 6.3 PeV energy region, the $\overline{\nu}_e$ flux becomes easy to measure, due to the enhanced rate of the "Glashow resonance" [1, 3, 28]. The reaction is $\overline{\nu}_e + e^- \rightarrow W^-$. Thus we also define the ratio

$$Q \equiv \frac{\Phi_{\overline{e}}}{\Phi_{\mathrm{tot}}}\,, \qquad (2.18)$$

where $\Phi_{\overline{e}}$ is the $\overline{\nu}_e$ flux.

In the rest of this paper, we will focus mostly on the T and R ratios. With the formulas we have given in section 2.1, all other flux ratios are easy to obtain.

2.2.1 Pion sources

Pion-sources present a initial neutrino flux ratios $\Phi_e^0 : \Phi_\mu^0 : \Phi_\tau^0 = 1 : 2 : 0$. It holds in this case that at Earth, $\Phi_e = \frac{1}{3}\Phi_{\text{tot}}(P_{ee} + 2P_{e\mu})$, $\Phi_\mu = \frac{1}{3}\Phi_{\text{tot}}(P_{e\mu} + 2P_{\mu\mu})$, and $\Phi_\tau = \frac{1}{3}\Phi_{\text{tot}}(P_{e\tau} + 2P_{\mu\tau})$, These simplify to

$$\text{pion sources:} \quad (\Phi_e : \Phi_\mu : \Phi_\tau) = \left(1 + 2\,\Delta : 1 - \Delta + \overline{\Delta}^2 : 1 - \Delta - \overline{\Delta}^2\right). \tag{2.19}$$

Here we have used the $(\Delta, \overline{\Delta}^2)$ expansion. The single (and small) terms containing only $|U_{e3}|^2$, which are present besides Δ and $\overline{\Delta}^2$ in eq. (2.9), drop out of these expressions. If one uses the (A, B, C) expansion, which includes ϵ' as well, then one finds that A drops out of the ratios, to leave:

$$\text{pion sources:} \quad (\Phi_e : \Phi_\mu : \Phi_\tau) = \left(1 + \frac{B}{9} : 1 - \frac{B}{18} + \frac{C}{9} : 1 - \frac{B}{18} - \frac{C}{9}\right). \tag{2.20}$$

These relations illustrate that deviations from the "canonical" $1 : 1 : 1$ result are of order $(\Delta, \overline{\Delta}^2)$, or $(\frac{1}{10}(B, C))$ and can therefore exceed 10%. Another result from these formulae is that the ratio

$$\frac{\Phi_\mu}{\Phi_\tau} \simeq 1 + 2\overline{\Delta}^2 = 1 + \frac{2}{9}C \tag{2.21}$$

is always larger than or equal to one. We checked that this is also true for the full expression. Hence, there cannot be more ν_τ than ν_μ at Earth if the initial flux composition is $1 : 2 : 0$.

Reorganizing the results in eq. (2.19), we get the ratios of our interest. They are[5]

$$T \simeq \frac{1}{3}(1 - \Delta + \overline{\Delta}^2) \quad \text{and} \quad R \simeq 1 + 3\,\Delta + \overline{\Delta}^2 + 3\,\Delta^2. \tag{2.22}$$

Numerically, using the full expressions, T lies between 0.32 and 0.39, while R ranges from 0.82 to 1.48. Therefore, deviations of more than 15% for T, and almost 50% for R can be expected.

The ratio of electron neutrinos to all other neutrino flavors is interesting in that it receives no quadratic corrections. The ratio is

$$\frac{\Phi_e}{\Phi_{\text{tot}}} \simeq \frac{1}{3}(1 + 2\,\Delta). \tag{2.23}$$

The tribimaximal values of the ratios T, R, and Φ_e/Φ_{tot} are clearly $\frac{1}{3}$, 1, and $\frac{1}{3}$, respectively.

2.2.2 Muon-damped sources

The fluxes at Earth for muon-damped sources are found to be

$$\begin{aligned}\text{muon-damped sources:} \quad (\Phi_e : \Phi_\mu : \Phi_\tau) &= (P_{e\mu} : P_{\mu\mu} : P_{\mu\tau}) \overset{\text{TBM}}{=} (4 : 7 : 7) \\ &\longrightarrow (4 - 2A + B : 7 + A - B + C : 7 + A - C),\end{aligned} \tag{2.24}$$

---

[5]The expression for R does not exactly follow from eq. (2.9), but is obtained by evaluating and expanding the full fraction. The difference is however negligible.

where we first insert the tribimaximal mixing values, and then show the result with corrections. From these, we get the ratios of interest:

$$T = P_{\mu\mu} \simeq \frac{1}{18}\,(7 + A - B + C) \simeq \frac{7}{18} - \Delta + \frac{1}{2}\,\overline{\Delta}^2 \,, \tag{2.25}$$

$$R = \frac{P_{e\mu}}{P_{\mu\tau}} \simeq \frac{4 - 2\,A + B}{7 + A - C} \simeq \frac{1}{7}\left(4 + 18\,\Delta + \frac{36}{7}\,\overline{\Delta}^2\right) \,,$$

where we have inserted on the far right-hand sides, the value $\sin^2\theta_{12} = \frac{1}{3}$. The quantities Δ and $\overline{\Delta}^2$ are therefore the quantities $\Delta_{\rm TBM}$ and $\overline{\Delta}^2_{\rm TBM}$ defined earlier in eq. (2.10). However, as mentioned below eq. (2.10), the differences are small. For exact tribimaximal mixing we have $T = \frac{7}{18}$ and $R = \frac{4}{7}$. Refs. [4] and [13] have proposed to use these sources to probe θ_{23} and the CP phase, respectively.

2.2.3 Neutron beam sources

Neutron beam sources have an initial $1:0:0$ flavor mix. We find for the flavor decomposition at Earth,

$$\text{neutron beam:} \quad (\Phi_e : \Phi_\mu : \Phi_\tau) = (P_{ee} : P_{e\mu} : P_{e\tau}) \overset{\rm TBM}{=} (5 : 2 : 2)$$
$$\longrightarrow (10 + 4\,A : 4 - 2\,A + B : 4 - 2\,A - B) \,. \tag{2.26}$$

Again, we give results for exact tribimaximal mixing, followed by the corrected expression. From these, we get the flavor ratios of interest:

$$T = P_{e\mu} \simeq \frac{1}{18}\,(4 - 2\,A + B) \simeq c_{12}^2\,s_{12}^2 + \Delta \simeq \frac{2}{9} + \Delta \,, \tag{2.27}$$

$$R = \frac{P_{ee}}{P_{e\tau}} \simeq \frac{10 + 4\,A}{4 - 2\,A - B} \simeq \frac{1 - 2\,c_{12}^2\,s_{12}^2}{c_{12}^2\,s_{12}^2}\left(1 + \frac{\Delta}{c_{12}^2\,s_{12}^2}\right) \simeq \frac{5}{2}\left(1 + \frac{9}{2}\,\Delta\right) \,.$$

Tribimaximal values are $\frac{2}{9}$ and $\frac{5}{2}$, respectively. Note that C, or alternatively $\overline{\Delta}^2$, which show up only in the μ–τ sector, do not appear in neutron beam formulae. The ratio of $\overline{\nu}_e$ to the total flux, initially unity, is simply $Q = P_{ee} \overset{\rm TBM}{=} \frac{5}{9} \rightarrow (10 + 4\,A)/18$.

2.2.4 Summarizing the flavor ratios

We summarize the situation for the flux ratios T and R. Tables 1, 2 and 3 show their ranges for the currently allowed 3σ ranges of the oscillation parameters. In the case of exact tribimaximal mixing, T is $1/3$, $7/18 \simeq 0.39$, and $2/9 \simeq 0.22$ for pion, muon-damped and neutron sources, respectively. If future neutrino oscillation experiments show that deviations from tribimaximal are small, then measurements of T with $\simeq 10\%$ precision would distinguish between pion and muon-damped sources. However, a low value of $T \sim 2/9$ would clearly indicate neutron sources.[6] On the other hand, currently allowed nonzero values of $|U_{e3}|$, ϵ or ϵ' lead to a possible overlap of the ratio T for all sources. This is shown in figure 4, where we display the allowed ranges of T for the 3σ ranges of the oscillation parameters. The range at the very left side of the plot is what is relevant to this discussion (the relevance of the rest of the plot concerns flux uncertainties, which we introduce in

	general case		TBM	
composition	$T = \Phi_\mu/\Phi_{\text{tot}}$	$R = \Phi_e/\Phi_\tau$	$T = \Phi_\mu/\Phi_{\text{tot}}$	$R = \Phi_e/\Phi_\tau$
$1 : 2 : 0$	$0.323 \div 0.389$	$0.818 \div 1.476$	0.333	1.000
$1 : 1.90 : 0.001$	$0.321 \div 0.386$	$0.834 \div 1.493$	0.331	1.017
$1 : 1.85 : 0.001$	$0.321 \div 0.384$	$0.842 \div 1.503$	0.330	1.026
$1 : 1.80 : 0.001$	$0.320 \div 0.382$	$0.850 \div 1.513$	0.329	1.036

Table 1: Pion sources: ranges of the ratios $T = \Phi_\mu/\Phi_{\text{tot}}$ and $R = \Phi_e/\Phi_\tau$ for the current 3σ ranges of the oscillation parameters. We used different flux compositions, pure $1 : 2 : 0$ and different impure cases. The values for tribimaximal mixing are also given.

	general case		TBM	
composition	$T = \Phi_\mu/\Phi_{\text{tot}}$	$R = \Phi_e/\Phi_\tau$	$T = \Phi_\mu/\Phi_{\text{tot}}$	$R = \Phi_e/\Phi_\tau$
$0 : 1 : 0$	$0.33 \div 0.51$	$0.34 \div 1.13$	0.39	0.57
$0.05 : 1 : 0$	$0.33 \div 0.49$	$0.40 \div 1.17$	0.38	0.63
$0.1 : 1 : 0$	$0.33 \div 0.48$	$0.46 \div 1.21$	0.37	0.68

Table 2: Muon damped sources: ranges of the ratios $T = \Phi_\mu/\Phi_{\text{tot}}$ and $R = \Phi_e/\Phi_\tau$ for the current 3σ ranges of the oscillation parameters. We used different flux compositions, pure $0 : 1 : 0$ and different impure cases. The values for tribimaximal mixing are also given.

	general case		TBM	
composition	$T = \Phi_\mu/\Phi_{\text{tot}}$	$R = \Phi_e/\Phi_\tau$	$T = \Phi_\mu/\Phi_{\text{tot}}$	$R = \Phi_e/\Phi_\tau$
$1 : 0 : 0$	$0.12 \div 0.35$	$1.39 \div 5.35$	0.22	2.50
$1 : 0.05 : 0$	$0.14 \div 0.35$	$1.35 \div 4.73$	0.23	2.34
$1 : 0.1 : 0$	$0.16 \div 0.35$	$1.32 \div 4.26$	0.24	2.21

Table 3: Neutron beam sources: ranges of the ratios $T = \Phi_\mu/\Phi_{\text{tot}}$ and $R = \Phi_e/\Phi_\tau$ for the current 3σ ranges of the oscillation parameters. We used different flux compositions, pure $1 : 0 : 0$ and different impure cases. The values for tribimaximal mixing are also given.

the next section). R, the ratio of electron to tau neutrinos, is for tribimaximal mixing 1, 4/7 and 5/2, respectively, for the pion, muon-damped, and neutron sources. Hence, much less precision would be required in order to distinguish the different source types. A large value of R would indicate a neutron source. However, the correction terms due to nonzero $|U_{e3}|$, ϵ and ϵ' are seen to have rather large pre-factors, so that again overlap can be expected. Results with these uncertainties are shown in figure 5. We show in figure 6 the ratio Q of $\overline{\nu}_e$ to the total flux. Recall that this ratio is important for the $\overline{\nu}_e + e^- \to W^-$ reaction in the ~ 6.3 PeV energy bin. With the Q observable, the neutron source ("cosmic β-beam") is clearly differentiated from the other source types. An interesting proposal [28]

---

[6]Alternatively, the same statements apply to the ratio $\Phi_\mu/(\Phi_e + \Phi_\tau)$, which is 1/2, $7/11 \simeq 0.64$ and $2/7 \simeq 0.29$.

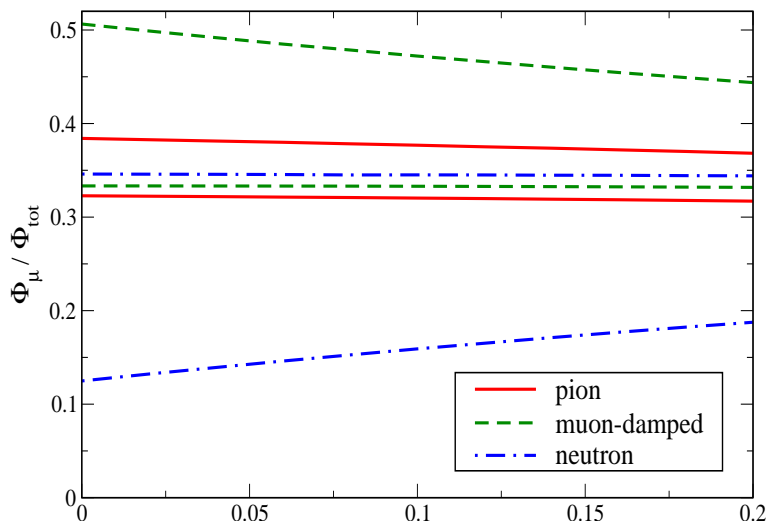**Figure 4:** The minimal and maximal values of the ratio T of muon neutrinos to the total flux, obtained by varying oscillation parameters within their 3σ ranges. The horizontal axis labels deviations (ζ or η) from the idealized flux compositions, parameterized as $1 : 2(1 - \zeta) : 0$ for pion sources, $\eta : 1 : 0$ for muon-damped sources, and $1 : \eta : 0$ for neutron beam sources.
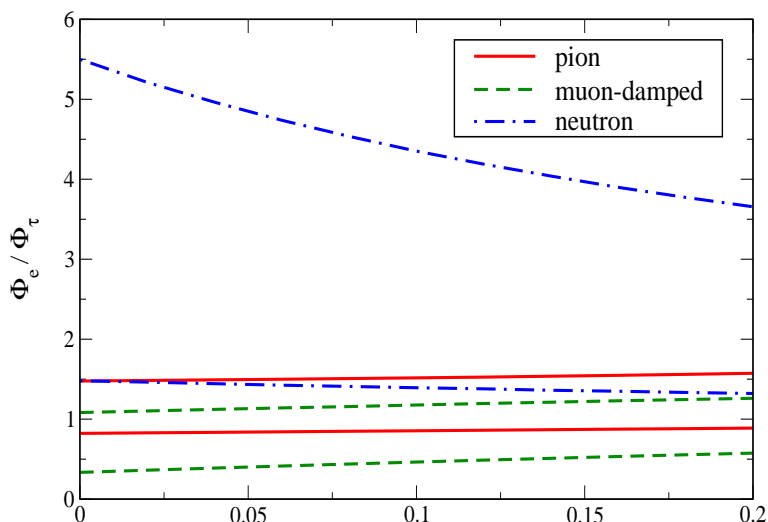

Figure 5: Same as previous figure for the ratio R of electron to tau neutrinos.

is that a measurement of Q as the ratio of the resonant $\overline{\nu}_e + e^- \rightarrow W^-$ 6.3 PeV bin to any off-resonance bin can differentiate between two nuances of the pion source, namely pp and $p\gamma$ beam dumps. The latter has no $\overline{\nu}_e$'s at production. Figure 6, in which neutrino luminosities are assumed equal for all source types, reveals that pp and $p\gamma$ are in principle distinguishable. In fact, it has been argued that the neutrino luminosity from pp is larger than that from $p\gamma$ by a factor ~ 2.4 (see [28] and references therein). This implies a better statistical determination of Q with the pp origin, and an additional signal for discriminating
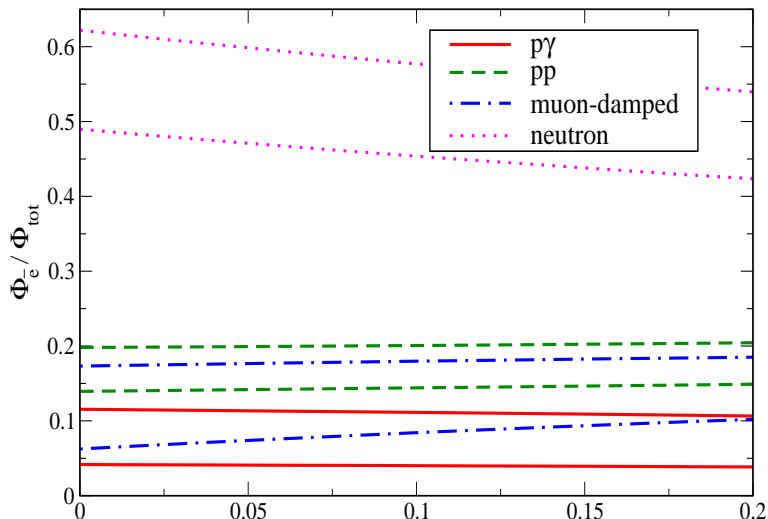
Figure 6: Same as previous figure for the ratio Q of $\overline{\nu}_e$ to all neutrinos.

pp and $p\gamma$. On the other hand, the muon-damped source also scales with this 2.4 factor, potentially confusing the discrimination between pp and $p\gamma$ sources. Also, it has been noted that the $\gamma\gamma \to \mu^+\mu^-$ reaction may provide some $\overline{\nu}_e$'s in a hot $p\gamma$ environment [40].

We note that for pion sources the ratios depend very little on θ_{12}. For the other sources, the dependence is stronger, as (unlike for pion sources) in the limit of $\epsilon = U_{e3} = 0$ the ratios depend on θ_{12}. In general, however, for equal deviations from the tribimaximal values, the dependence on θ_{12} is weaker than the dependences on θ_{23} and $|U_{e3}| \cos\delta$. Nevertheless, we note that schematically one may write

$$\text{Ratio(pion)} = c_0 + c_1\,\Delta + c_2\,\overline{\Delta}^2 \ , \qquad (2.28)$$
$$\text{Ratio(muon-damped)} = f_0(\theta_{12}) + f_1(\theta_{12})\,\Delta + f_2(\theta_{12})\,\overline{\Delta}^2 \ ,$$
$$\text{Ratio(neutron)} = g_0(\theta_{12}) + g_1(\theta_{12})\,\Delta \ .$$

The zeroth order expression for pion sources does not depend on θ_{12}. In addition, typically we have that the magnitude of $c_0/c_{1,2}$ is larger than the magnitudes of $f_0(\theta_{12})/f_{1,2}(\theta_{12})$ and $g_0(\theta_{12})/g_{1,2}(\theta_{12})$. For instance, for the ratio T, we find from the above that $|c_0/c_1| = |c_0/c_2| = 1$, whereas $|f_0/f_1| = |f_0/f_2| = \frac{1}{2}\,(1 - c_{12}^2\,s_{12}^2) \simeq 7/18$ and $|g_0/g_1| = c_{12}^2\,s_{12}^2 \simeq 2/9$. Therefore, the ratios for muon-damped and neutron sources are significantly more sensitive to Δ (and $\overline{\Delta}^2$), and thus on deviations from vanishing U_{e3} and maximal θ_{23}. This simple fact is the reason that flux ratios of muon-damped and neutron beam sources are better suited to probe such deviations. Noting further that Δ and $\overline{\Delta}^2$ depend strongly on θ_{23}, we can expect that neutrino telescopes will be most sensitive to the θ_{23} parameter.

3. Uncertainties in initial flavor composition

Up to this point we have obtained expressions for the flux ratios in the cases of exact initial flavor composition. However, as indicated in the Introduction, one expects on general

grounds some deviations from the idealized $1 : 2 : 0$ (pion source), $0 : 1 : 0$ (muon-damped source), or $1 : 0 : 0$ (neutron source) flavor ratios. The implications of non-idealized source ratios, on the extraction of the neutrino mixing parameters, have not been previously studied in detail. In this section, we will first estimate the amount of "impurity" in the initial flux compositions. After that, we will discuss some examples leading to incorrect inferences if care is not taken.

3.1 Estimating realistic flavor-flux compositions

When the source of neutrinos is a hadronic beam dump which produces pions and consequently neutrinos from $\pi \to \mu\,\nu_\mu \to e\,\nu_\mu\,\nu_\mu\,\nu_e$, the pion decay chain, naive counting gives a flux ratio for ν_e to ν_μ of $1 : 2$ and no ν_τ. However, the "wrong-helicity" muon polarization from pion decay makes the ν_μ from its decay softer, thus reducing the effective ν_μ count. For a canonical spectral index of $\alpha = 2$, the predicted initial flavor ratio is $1 : 1.86$ (see the detailed discussion in ref. [34]). In addition, the production and decay of kaons also produces neutrinos. The kaon decay modes which produce charged pions in the final state (e.g., $K_S \to 2\pi$, $K^\pm \to 2\pi$, $K_L \to 3\pi$) give rise to the same ratio of $1 : 1.86$.

We now extend the analysis of ref. [34] to include leptonic and semi-leptonic K decays, and production and decay of heavy flavors. The appreciable $K^\pm \to \mu^\pm + \overset{(-)}{\nu_\mu}$ chain and the three-body semi-leptonic decay modes of $K^\pm$ and especially K_L also produce neutrinos. The $K^\pm \to \mu^\pm + \overset{(-)}{\nu_\mu}$ chain has a $\nu_e : \nu_\mu$ ratio of $1 : 2.8$ and the K_L semi-leptonic decays give a ratio of $1 : 0.75$. We use a K/π ratio of 0.15 in our estimates, although the dependence on this ratio is rather weak. When all the modes are added to the pion chain neutrinos, and all the branching ratios and neutrino energy spectra are taken into account (following the methods described in [34]), the final flavor mix ends up being surprisingly close to the value $1 : 1.86$, which was obtained by including only the pion chain. Finally, at high energies, production of the heavy flavors c and b is expected. The semi-leptonic decay of these heavy flavors gives rise to so-called "prompt" neutrinos, characterized by the flux ratio for $\nu_e : \nu_\mu$ of $1 : 1$. There is also a small flux of ν_τ from the decay mode $D_s \to \tau\,\nu_\tau$, which has a branching ratio of about 6%. Bottom quark decays also produce some ν_τ, but b-production is down compared to c-production by a factor of about 30.

After taking all cross-sections, decaying species, branching ratios, and neutrino energy distributions into account, our final estimate of the corrected flavor mix is

$$\Phi_e^0 : \Phi_\mu^0 : \Phi_\tau^0 = 1.00 : 1.852 : 0.001 \ . \tag{3.1}$$

Remarkably, the $\nu_e : \nu_\mu$ ratio remains very close to the original ratio $1 : 1.86$ due to just pion decay. Notice that the ν_τ content is a negligible 0.1% at most. The main source of uncertainty in our estimate is the spectral index at injection, α. Figure 3 of ref. [34] shows that the ratio $1 : 1.86$ is modified by a few percent to $1 : 1.9$ if $\alpha = 1.7$, and to $1 : 1.8$ if $\alpha = 2.3$.

We quantify the deviations with the single parameter ζ:

$$\Phi_e^0 : \Phi_\mu^0 : \Phi_\tau^0 = 1 : 2\,(1 - \zeta) : 0, \tag{3.2}$$

and we can expect ζ to be ~ 0.1. The ratio $1 : 1.86$ for electron to muon neutrinos corresponds to $\zeta = 0.07$.

Damped muon sources result when muon energy loss mechanisms are operative. The effects of muon energy loss for the flavor mix have been discussed by a number of authors [29 – 34, 14] for a number of sources (such as gamma ray bursts), and models (such as Waxman-Bahcall type). From refs. [31, 34, 14] one can draw two general conclusions: first, the onset of muon-damping happens abruptly in energy. In ref. [14], the onset is within a factor of 2 to 3 of 10^8 GeV. Second, the ν_e flux may be severely suppressed, but it never goes to zero. It never falls below 4% in ref. [14], and never below 2% in ref. [34]. Consequently, we parameterize the initial muon-damped sources as $(\eta : 1 : 0)$, where η is at least a few percent.

Finally, for the neutron sources a pion pollution of order 10% has been estimated in ref. [4]. So for the muon-damped and neutron sources, we introduce the single parameter η and use for the initial flux ratios

$$\Phi_e^0 : \Phi_\mu^0 : \Phi_\tau^0 = \eta : 1 : 0 \quad \text{(muon-damped)}, \quad \text{and} \quad 1 : \eta : 0 \quad \text{(neutron)}. \qquad (3.3)$$

For typical values of ζ and η, we show in tables 1, 2 and 3 the ranges of the flux ratios under consideration. The oscillation parameters are varied in their currently allowed 3σ ranges. Results for pion sources are the most stable. We also give in the tables the values of the flux ratios if the neutrino parameters are fixed to their tribimaximal values. One concludes that impure initial flux compositions should be taken into account when discussing the prospects of inferring neutrino parameters with flux ratio measurements.

It is worth mentioning that another possible source of deviations from idealized flux ratios is new physics. For example, $3 + 2$ sterile neutrino scenarios [41, 42] (still allowed even after the MiniBooNE results [43]), can cause deviations of the flux ratios of order 10%, thereby interfering with the program of inferring deviations from three-flavor tribimaximal mixing [44, 42].

We note that a different parameterization for the initial flavor ratios is given in ref. [6]. There the unit normalization of the sum of the flavor ratios is emphasized by introducing two polar angles for the vector on the unit sphere, $\Phi_e : \Phi_\mu : \Phi_\tau = \sin^2 \xi \cos^2 \zeta : \cos^2 \xi \cos^2 \zeta : \sin^2 \zeta$. Although this parameterization (the angular ζ-parameter of ref. [6] is not to be confused with our small ζ-parameter defined in eq. (3.2)) and ours are equivalent, ours does have the advantage that the introduced parameters in eqs. (3.2) and (3.3) are small, thereby allowing the perturbative expansions we present in the next subsections. Also, the focuses in ref. [6] and in our work are different. The former emphasizes determination of the initial flux composition, i.e., determining ξ and ζ, when the neutrino mixing parameters are known with sufficient precision. In our work we include the influence of uncertainties from both the initial fluxes and the neutrino mixing parameters upon the experimental program, with particular attention paid to the consequences for extraction of precise neutrino parameters.

3.2 Parameterized pion sources

Justifiably neglecting the ν_τ contribution, the initial flux composition from pion sources is

$\Phi_e^0 : \Phi_\mu^0 : \Phi_\tau^0 = 1 : 2 \, (1 - \zeta) : 0$. In the limit of $\zeta = 0$, we recover the ratios of section 2.2. The detectable fluxes behave according to

$$\Phi_e \propto P_{ee} + 2 \, (1 - \zeta) \, P_{e\mu} = 1 - 2 \, \zeta \, c_{12}^2 \, s_{12}^2 + 2 \, \Delta \, (1 - \zeta) - 2 \, \zeta \, (1 - 2 \, c_{12}^2 \, s_{12}^2) \, |U_{e3}|^2 \,,$$
$$\Phi_\mu \propto P_{e\mu} + 2 \, (1 - \zeta) \, P_{\mu\mu} = 1 - \Delta - \zeta \, (1 - c_{12}^2 \, s_{12}^2) + \overline{\Delta}^2 + 2 \, \zeta \, \Delta + \zeta \, (1 - 2 \, c_{12}^2 \, s_{12}^2) \, |U_{e3}|^2 \,,$$
$$\Phi_\tau \propto P_{e\tau} + 2 \, (1 - \zeta) \, P_{\mu\tau} = 1 - \Delta - \zeta \, (1 - c_{12}^2 \, s_{12}^2) - \overline{\Delta}^2 + \zeta \, (1 - 2 \, c_{12}^2 \, s_{12}^2) \, |U_{e3}|^2 \,,$$

plus cubic terms in the small parameters. With the alternate (A, B, C) expansion, we have

$$\begin{aligned}
\Phi_e &\propto 1 + \frac{B}{9} - 2 \zeta \, (4 - 2 \, A + B) \,, \\
\Phi_\mu &\propto 1 - \frac{B}{18} + \frac{C}{9} - 2 \zeta \, (7 + A - B + C) \,, \\
\Phi_\tau &\propto 1 - \frac{B}{18} - \frac{C}{9} - 2 \zeta \, (7 + A - C) \,.
\end{aligned} \tag{3.4}$$

We show in figure 4 the allowed range of the ratio T for pion sources as a function of ζ, while figure 5 shows the same for R, and figure 6 the same for Q. The largest effect is seen in the ratio Q, in the case of a $p\gamma$ neutrino source.

Impure sources will still lead to deviations from "pure" values of the flux ratios, even if neutrino mixing is exactly tribimaximal:

$$T_{\mathrm{TBM}} = \frac{9 - 7\zeta}{27 - 18\zeta} \simeq \frac{1}{3} \left(1 - \frac{\zeta}{9} \right) \quad \text{and} \quad R_{\mathrm{TBM}} = \frac{9 - 4\zeta}{9 - 7\zeta} \simeq 1 + \frac{\zeta}{3} \,. \tag{3.5}$$

The deviation is seen to be stronger in R_{TBM}.

Possible nonzero θ_{13} and non-maximal θ_{23} may be compensated by the "impurity factor" ζ. The flux ratio $\Phi_e/\Phi_{\mathrm{tot}}$ illustrates this "confusion". From eq. (2.23) we know that for $\zeta = 0$ the ratio is given by $\frac{1}{3}(1 + 2\,\Delta)$. From eq. (3.4), it is (neglecting the very small single terms depending on $|U_{e3}|^2$) easily obtained that a nonzero Δ is compensated by an initial flux uncertainty if the following relation holds:

$$\zeta = -\frac{3 \, \Delta}{1 - 3 \, c_{12}^2 \, s_{12}^2} \simeq -9 \, \Delta \,. \tag{3.6}$$

This value of uncertainty returns the ratio to the value $\frac{1}{3}$, even though $\Delta \neq 0$. The ratio of muon and tau neutrino fluxes provides another example. Compensation occurs if

$$\zeta = -\frac{\Delta}{\overline{\Delta}^2} \,. \tag{3.7}$$

In figure 7 we show the distribution of $|U_{e3}| \cos\delta$ against $\sin^2 \theta_{23}$ if the flux ratio T is measured to be $\frac{1}{3}$ and 0.35. We choose a pure source with $1 : 2 : 0$ and three different impure sources motivated by our estimates in section 3.1. Figure 8 shows the same for $R = 1$ and $R = 1.1$. It is obvious that the covered area in parameter space changes considerably when the flux composition is varied. Table 1 gives the numerical values of the ranges. Though the upper and lower limits are barely modified by varying ζ, there are easily situations in which nonzero ζ is dramatic. For example, if T was measured to
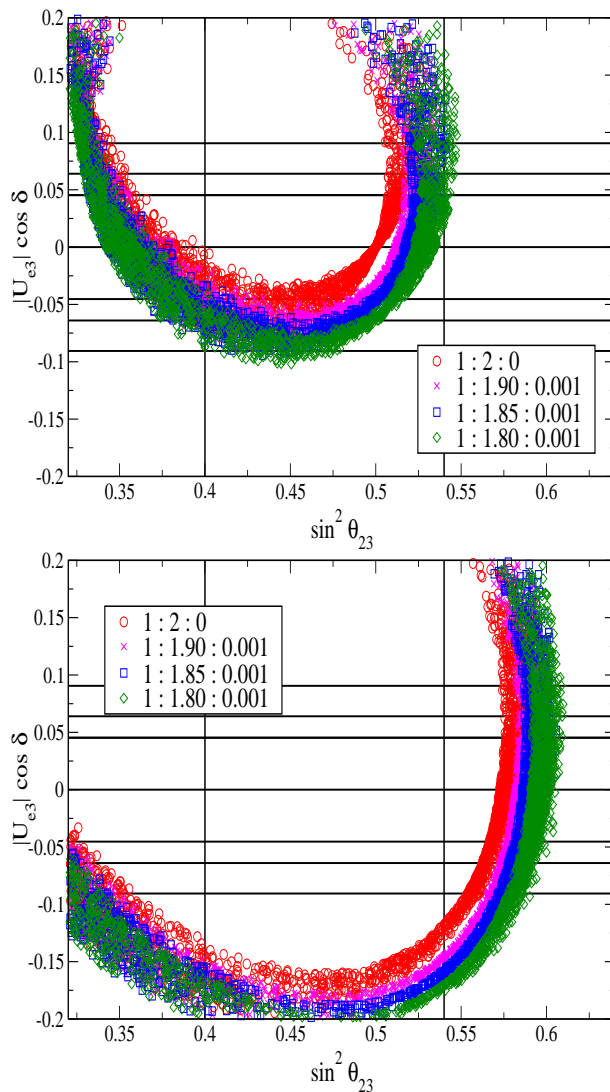
Figure 7: Distribution of $|U_{e3}| \cos \delta$ against $\sin^2 \theta_{23}$ if the flux ratio $\Phi_\mu/\Phi_{\text{tot}}$ is measured to be $\frac{1}{3}$ (top) and 0.35 (bottom), for different initial flavor compositions. The red circles are for pure $1 : 2 : 0$, the magenta crosses are for $1 : 1.90 : 0.001$, the blue squares are for $1 : 1.85 : 0.001$ and the green diamonds are for $1 : 1.80 : 0.001$. Indicated also is the allowed 1σ range of θ_{23} and of $|U_{e3}| \cos \delta$ for (from top to bottom above zero) $\delta = 0$, $\delta = \pi/4$ and $\delta = \pi/3$. The value $\delta = \pi/2$ means $|U_{e3}| \cos \delta = 0$.

be $\frac{1}{3}$ and it was known that $\sin^2 \theta_{23} = \frac{1}{2}$, then one would infer for an ideal pion source that $|U_{e3}| \cos \delta = 0$. If in addition $|U_{e3}|$ was known to be nonzero, then δ would be $\pi/2$ and CP-violation would be inferred. However, if instead, the true value is $\zeta = 0.1$, then $|U_{e3}| \cos \delta \simeq 0.06$ would hold, and if $|U_{e3}|$ was known to be 0.06 then CP may or may not be broken.

The simple examples given here show that care has to be taken when conclusions about neutrino mixing parameters are drawn from flux ratio measurements.
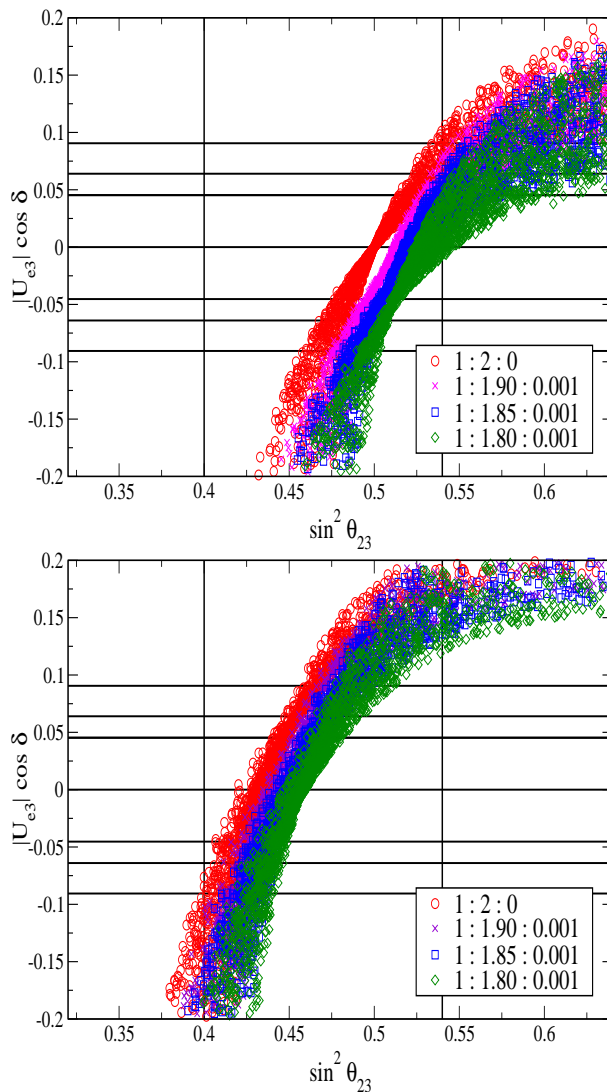
Figure 8: Same as previous figure for the flux ratio Φ_e/Φ_τ assumed to be 1 (top) and 1.1 (bottom).

3.3 Parameterized muon-damped sources

Here we investigate the impact of the non-idealized initial muon-damped source ratios $\Phi_e^0 : \Phi_\mu^0 : \Phi_\tau^0 = \eta : 1 : 0$. Figures 4, 5 and 6 show the minimal and maximal values of the ratios T, R and Q. In general, the dependence on initial flavor deviations is larger here than it was with pion sources. The fluxes behave according to

$$\Phi_e \propto P_{e\mu} + \eta\, P_{ee} = \frac{1}{18}\left(4 - 2A + B + 2\eta\left(5 + 2A\right)\right),$$

$$\Phi_\mu \propto P_{\mu\mu} + \eta\, P_{e\mu} = \frac{1}{18}\left(7 + A - B + C + \eta\left(4 - 2A + B\right)\right), \qquad (3.8)$$

$$\Phi_\tau \propto P_{\mu\tau} + \eta\, P_{e\tau} = \frac{1}{18}\left(7 + A - C + \eta\left(4 - 2A - B\right)\right).$$

Table 2 shows the numerical effect of $\eta \neq 0$ for neutrino fluxes from muon-damped sources.
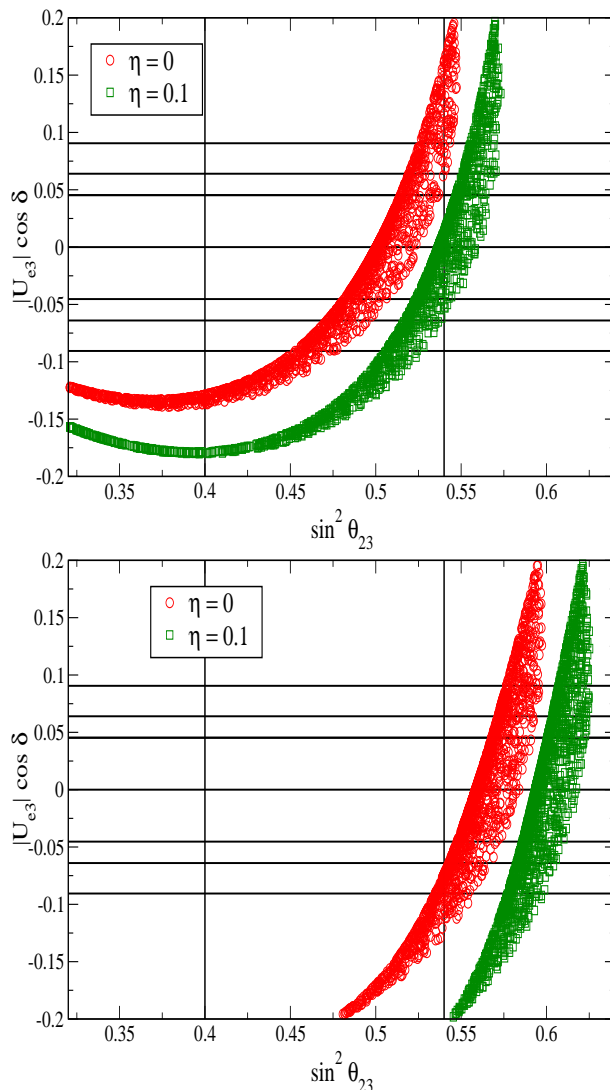
Figure 9: Distribution of $|U_{e3}| \cos \delta$ against $\sin^2 \theta_{23}$ if the flux ratio $\Phi_\mu/\Phi_{\text{tot}}$ is measured to be 7/18 (top) and 0.42 (bottom); the initial flavor mix is $\eta : 1 : 0$, and $\sin^2 \theta_{12} = \frac{1}{3}$ is assumed.

Comparing with table 1 for pion sources, one sees that the effect of source uncertainty is larger for muon-damped sources compared to pion sources.

If neutrinos mix tribimaximally, then

$$T_{\text{TBM}} = \frac{1}{18} \frac{7+4\,\eta}{1+\eta} \simeq \frac{1}{18}\,(7-3\,\eta) \quad \text{and} \quad R_{\text{TBM}} = \frac{4+10\,\eta}{7+4\,\eta} \simeq \frac{1}{7}\left(4 + \frac{54}{7}\,\eta\right) . \qquad (3.9)$$

As with pion sources, the parameter dependence is stronger for R_{TBM} than T_{TBM}. Figure 9 displays the distribution of $|U_{e3}| \cos \delta$ against $\sin^2 \theta_{23}$, taking T to be 7/18 and 0.42, and for simplicity, fixing $\sin^2 \theta_{12} = \frac{1}{3}$. For the two chosen values of $\eta = 0$ and 0.1, the allowed areas do not meet, which shows again that the sensitivity on impure initial fluxes is stronger for muon-damped than for pion sources. Moreover, the dependence on the actual value of the
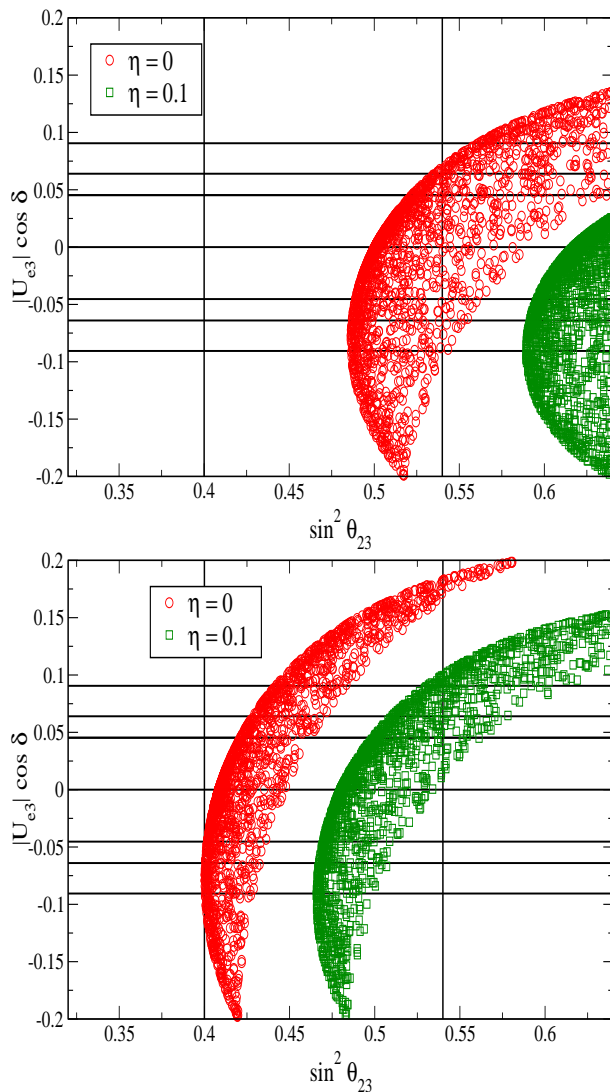
Figure 10: Same as previous figure for the ratio of electron to tau neutrinos measured to be 7/18 (top) and 0.7 (bottom).

ratio T is weaker than the dependence on ζ. In figure 10 we show correlated dependences, fixing the ratio R to be 4/7 and 0.7.

Let us discuss a hypothetical but illustrative example. If all neutrino parameters but δ were known exactly, and if T were measured without any uncertainty, then the value of $\cos\delta$ can be extracted [4, 13]. In figure 11 we display this possibility, taking optimistic values of the other parameters. We assume a large value of θ_{13} and maximal θ_{23} in order to maximize the dependence on $\cos\delta$. Suppose now that $T = 0.397$ were measured. Assuming that $\eta = 0$, i.e., a very pure muon-damped source, one would conclude that $\cos\delta = -0.5$, thereby inferring leptonic CP-violation. However, if in reality $\eta = 0.1$, then the same $T_\mu = 0.397$ value would instead mean that $\cos\delta = -1$ and CP is conserved.

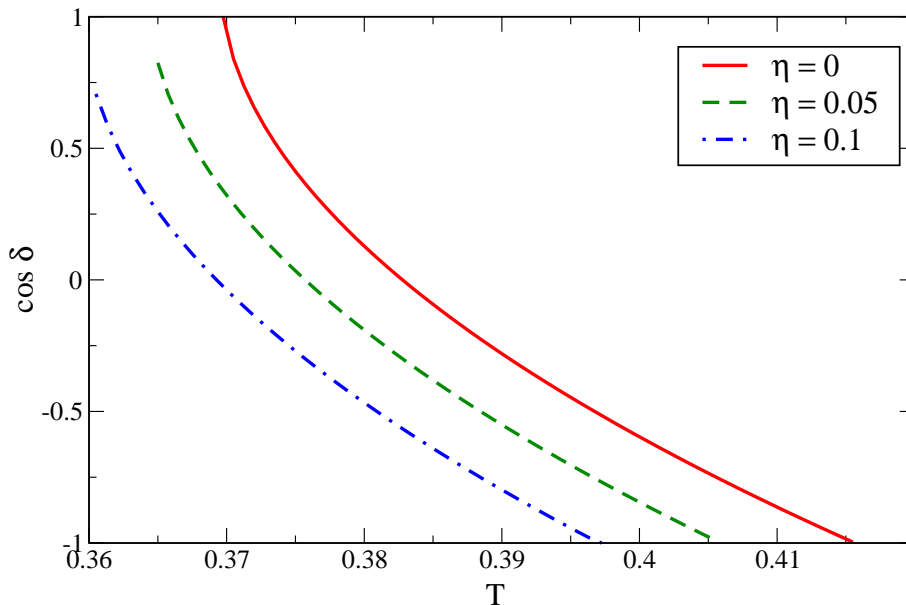Let us present another example using figure 12. In figure 12 the dependence of T on

Figure 11: Extracted value of $\cos\delta$ from an exact measurement of the ratio T of muon neutrinos to the total flux for an initial flux composition $\eta : 1 : 0$. The oscillation parameters are fixed to $\theta_{23} = \pi/4$, $\sin^2\theta_{12} = \frac{1}{3}$, and $|U_{e3}| = 0.15$.
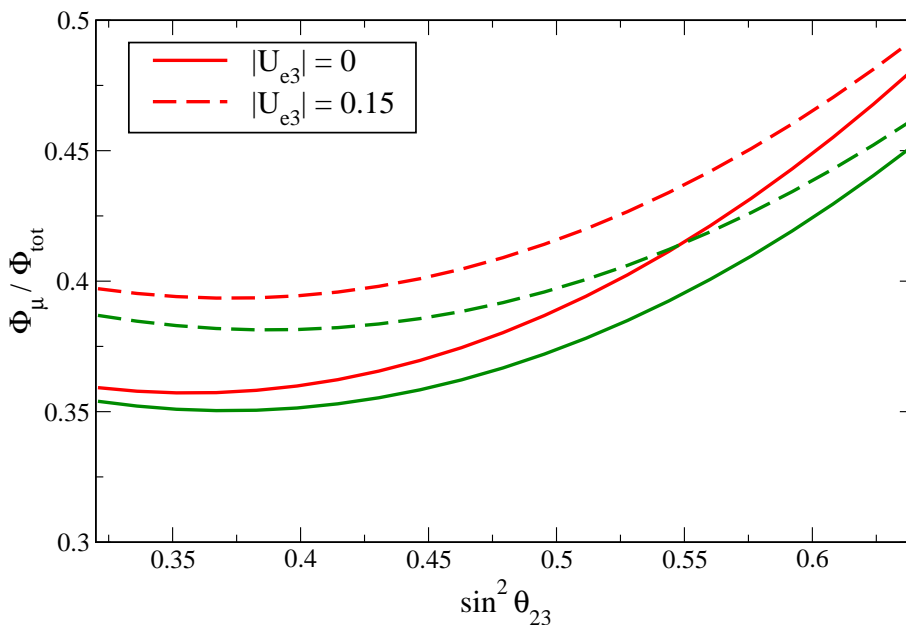

Figure 12: Dependence on $\sin^2\theta_{23}$ of the ratio of muon neutrinos to the total flux. The red lines are for $0 : 1 : 0$, while the green lines below are for $0.1 : 1 : 0$. We have chosen $\delta = \pi$ and $\sin^2\theta_{12} = \frac{1}{3}$.

$\sin^2 \theta_{23}$ [5] is shown, for two values of $|U_{e3}|$ and for $\eta = 0$ and 0.1. The nonzero η introduces a few percent uncertainty in the flux ratio, especially for $\sin^2 \theta_{23} \geq 0.5$. In general, the ratio decreases. The curves for different $|U_{e3}|$ do not meet when $\eta = 0$; however, for $\eta = 0.1$ they cross each other, thereby destroying the possibility to disentangle the two chosen values of $|U_{e3}|$.

3.4 Parameterized neutron beam source

Finally we come to the impact of an initial β-beam source which may not be pure, i.e., $\Phi_e^0 : \Phi_\mu^0 : \Phi_\tau^0 = 1 : \eta : 0$. In this case the flux ratios at Earth are given by

$$
\begin{aligned}
\Phi_e &\propto P_{ee} + \eta \, P_{e\mu} = \frac{1}{18} \left(10 + 4 \, A + \eta \, (4 - 2 \, A + B) \right) , \\
\Phi_\mu &\propto P_{e\mu} + \eta \, P_{\mu\mu} = \frac{1}{18} \left(4 - 2 \, A + B + \eta \, (7 + A - B + C) \right) , \\
\Phi_\tau &\propto P_{e\tau} + \eta \, P_{\mu\tau} = \frac{1}{18} \left(4 - 2 \, A - B + \eta \, (7 + A - C) \right) .
\end{aligned}
\tag{3.10}
$$

The tribimaximal values for oscillation parameters leads to

$$
T_{\text{TBM}} = \frac{1}{18} \frac{4 + 7\eta}{1 + \eta} \simeq \frac{1}{9} \left(2 + \frac{3}{2} \eta \right) \quad \text{and} \quad R_{\text{TBM}} = \frac{10 + 4\eta}{4 + 7\eta} \simeq \frac{5}{2} \left(1 - \frac{27}{20} \eta \right) .
\tag{3.11}
$$

More generally, we show in figures 4, 5 and 6 the minimal and maximal values of the ratios T, R and Q (flux composition $\nu_e : \overline{\nu}_e : \nu_\mu : \overline{\nu}_\mu : \nu_\tau : \overline{\nu}_\tau = 0 : 1 : \eta/2 : \eta/2 : 0 : 0$) that result when the oscillation parameters are allowed to vary over their 3σ ranges. As was the case with muon-damped sources, deviations from pure flux compositions have more impact on neutron sources than on pion sources. And again, the dependence on the actual value of the ratio T or R is weaker than the dependence on ζ. Table 3 confirms these remarks. In figures 13 and 14 the distributions of $|U_{e3}| \cos \delta$ against $\sin^2 \theta_{23}$ for two characteristic values of T (2/9 and 0.26) and R (5/2 and 2) are given.

Recall that Δ but not $\overline{\Delta}^2$ appears in the flux ratios for neutron sources. This makes it possible to give a simple formula for the special case where the effect of nonzero Δ is exactly compensated by a nonzero η. If

$$
\eta = 2 \frac{\Delta + (1 - 2 \, c_{12}^2 \, s_{12}^2) \, |U_{e3}|^2}{1 - 3 \, c_{12}^2 \, s_{12}^2} \simeq 6 \left(\Delta + \frac{5}{9} \, |U_{e3}|^2 \right) ,
\tag{3.12}
$$

then the tribimaximal value $T = c_{12}^2 \, s_{12}^2$ necessarily results. We give in figure 15 an example of the dependence of T on $|U_{e3}|$ and θ_{23}. It can be seen that impure initial flux compositions can influence statements on the octant of θ_{23}. For instance, measuring $T = 0.2$ would rule out $\sin^2 \theta_{23} > 0.5$ only if $\eta = 0$. However, if $\eta = 0.1$, then $T = 0.2$ is compatible with $\sin^2 \theta_{23} = 0.55$, and the octant of θ_{23} is different from the one inferred if $\eta = 0$.

4. Summary and conclusions

We have considered in this paper neutrino mixing and flux ratios of astrophysical neutrinos. We first have expanded the expressions in terms of small parameters $\epsilon = \pi/4 - \theta_{23}$ and
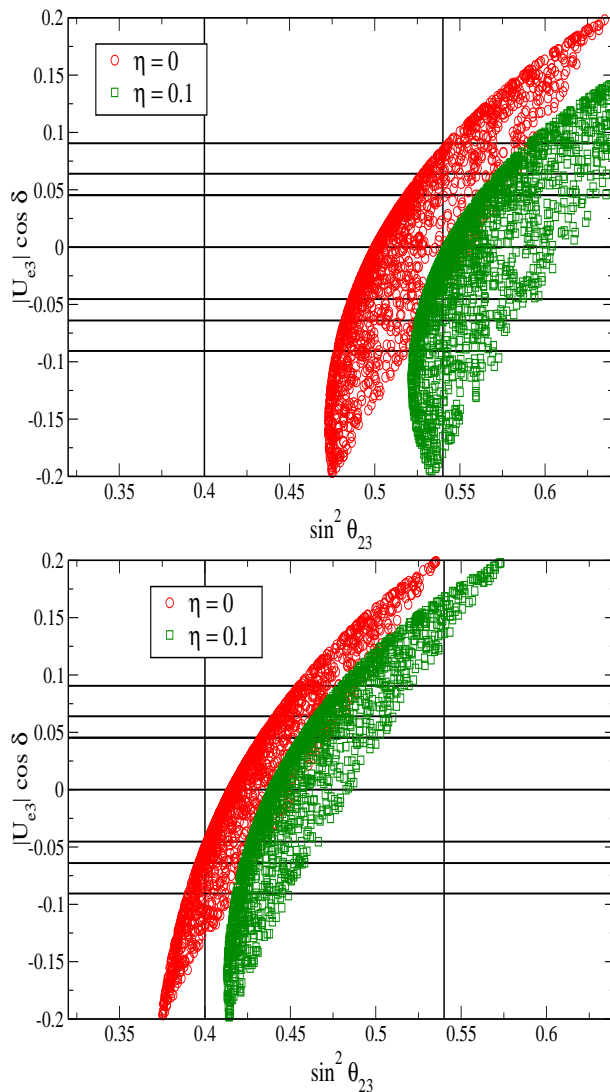
Figure 13: Distribution of $|U_{e3}| \cos \delta$ against $\sin^2 \theta_{23}$ if the flux ratio $\Phi_\mu / \Phi_{\rm tot}$ is measured to be $2/9$ (top) and 0.26 (bottom), for an initial flavor mix of $1 : \eta : 0$.

$|U_{e3}|$, while leaving θ_{12} free. The small parameters ϵ and $\Re\{U_{e3}\}$ measure the symmetry breaking of $\nu_\mu \leftrightarrow \nu_\tau$ interchange symmetry. With this expansion, we showed that the first and second order corrections which characterize the deviations from μ–τ symmetry, Δ and $\overline{\Delta}^2$, appear universally. The universal corrections Δ and $\overline{\Delta}^2$ are given in eqs. (2.6), (2.7). Each can take values as large as 0.1. Compact results for the mixing probabilities, in terms of Δ and $\overline{\Delta}^2$, are shown in eqs. (2.5), (2.9).

The second order term $\overline{\Delta}^2$ appears only in the μ–τ sector (therefore it is not relevant for flux ratios from neutron beam sources) and is positive semidefinite. Because it can exceed the first order term, it is necessary to include it in analytical considerations. It vanishes only for $\epsilon = |U_{e3}| \cos \delta = 0$, whereas the first order term can vanish also for nonzero values of ϵ and $|U_{e3}|$.
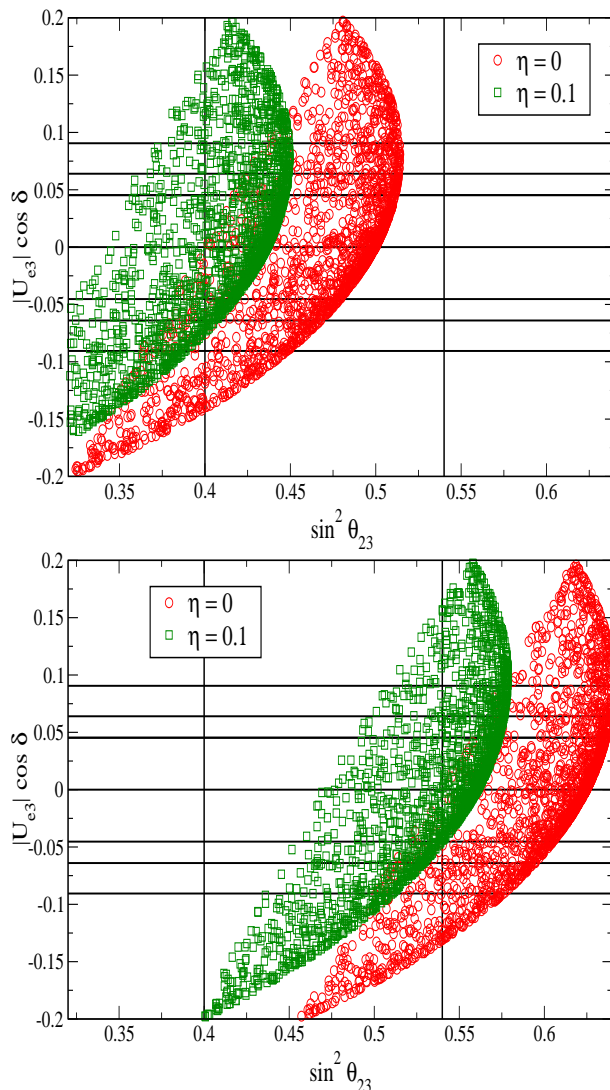
Figure 14: Same as previous figure for the ratio of electron to tau neutrinos measured to be 5/2 (top) and 2 (bottom).

In general, if the initial flavor mix is exactly $1 : 2 : 0$, then neutrino mixing transforms these ratios to $(\Phi_e : \Phi_\mu : \Phi_\tau) = (1 + 2\,\Delta) : \left(1 - \Delta + \overline{\Delta}^2\right) : \left(1 - \Delta - \overline{\Delta}^2\right)$. Hence, there are always more muon than tau neutrinos upon arrival at Earth. The ratio of muon neutrinos to all neutrinos can deviate by more than 15% from the tribimaximal value $\frac{1}{3}$, while the ratio of electron to tau neutrinos can deviate by up to 50% from the tribimaximal value 1.

As the solar neutrino mixing parameter $\sin^2 \theta_{12}$ is close to the tribimaximal value $\frac{1}{3}$, we next included $\epsilon' = \arcsin \sqrt{\frac{1}{3}} - \theta_{12}$ in our set of expansion parameters. With this expansion set, three universal corrections A, B and C, defined in [37] and reproduced here in eq. (2.15), characterize the deviations from tribimaximal mixing. Very concise
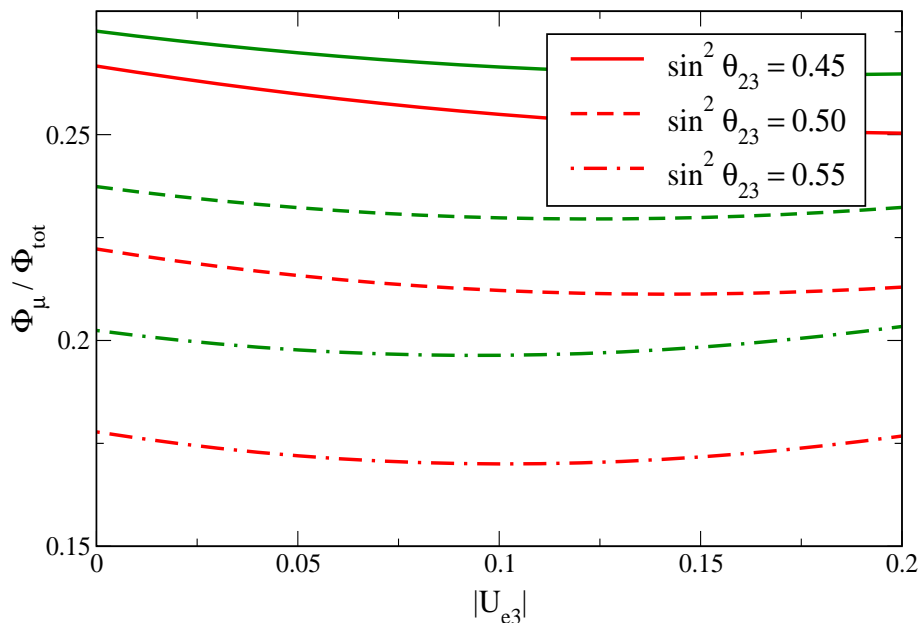
Figure 15: Dependence on θ_{13} of the ratio of muon neutrinos to the total flux. The red lines are for $1:0:0$, while the green lines above are for $1:0.1:0$. We have chosen $\delta = \pi$ and $\sin^2 \theta_{12} = \frac{1}{3}$.

expressions for the mixing probabilities result [37], as seen in eqs. (2.14).

In the second part of this paper, we investigated the purity of initial neutrino-flavor ratios expected from three types of astrophysical sources. The initial flavor ratios commonly considered in the literature are

(i) pion sources (with a complete pion decay chain), having initial ratios $1:2:0$;

(ii) muon-damped sources (initial pions but an incomplete decay chain), having initial ratios $0:1:0$;

(iii) neutron beam sources (so-called "cosmic β-beams"), with initial neutrino flavor ratios $1:0:0$.

The idealized flavor ratios of all three source types are subject to small but important corrections. We investigated the effects of realistic corrections on the extraction of neutrino parameters from measurements of flux ratios. We found that the muon-damped and neutron beam sources are more sensitive to initial flavor deviations than is the pion source. In addition, the muon-damped and neutron beam sources are also more sensitive to deviations of oscillation parameters θ_{23} and U_{e3} from $\pi/4$ and zero, respectively. These sensitivities can be easily seen by considering the ratio T of muon neutrinos to the total flux. For $\sin^2 \theta_{12} = \frac{1}{3}$, this ratio reads

$$T = \frac{\Phi_\mu}{\Phi_{\rm tot}} \simeq \begin{cases} \frac{1}{3}\left(1 - \Delta + \overline{\Delta}^2 - \frac{1}{9}\zeta\right), & \text{pion source} \quad (1:2\,(1-\zeta):0)\,, \\ \frac{7}{18} - \Delta + \frac{1}{2}\overline{\Delta}^2 - \frac{1}{6}\eta, & \text{muon-damped source} \quad (\eta:1:0)\,, \\ \frac{2}{9} + \Delta + \frac{1}{6}\eta, & \text{neutron beam source} \quad (1:\eta:0)\,, \end{cases}$$

where the parameterized initial flavor-ratios are $1 : 2(1 - \zeta) : 0$, $\eta : 1 : 0$ and $1 : \eta : 0$, respectively. For pion sources, the zeroth order expression is $T = \frac{1}{3}$, and with $|\Delta, \overline{\Delta}^2, \zeta, \eta| \lesssim 0.1$, deviations can be up to 15% for oscillation-induced effects and of order 1% for impure flavor mixes. For muon-damped sources, on the other hand, the effect of uncertain oscillation parameters is up to 30%, and the effect of nonzero η is more than 5%. The effect on neutron beam sources may be even more dramatic: the observable T can change by more than 50% due to deviations from tribimaximality, and by order 10% due to impurities. We have also considered the ratio R of electron to tau neutrinos, which in the same $\sin^2 \theta_{12} = 1/3$ limit reads

$$
R = \frac{\Phi_e}{\Phi_\tau} \simeq \begin{cases} 1 + 3\,\Delta + \overline{\Delta}^2 + \frac{\zeta}{3}\,, & \text{pion source} \quad (1 : 2\,(1 - \zeta) : 0)\,, \\ \frac{4}{7}\left(1 + 18\,\Delta + \frac{36}{7}\,\overline{\Delta}^2 + \frac{54}{7}\,\eta\right)\,, & \text{muon-damped source} \quad (\eta : 1 : 0)\,, \\ \frac{5}{2}\left(1 + \frac{9}{2}\,\Delta - \frac{27}{20}\,\eta\right)\,, & \text{neutron beam source} \quad (1 : \eta : 0)\,. \end{cases}
$$

The magnitude of coefficients reveal that the effects of nonzero Δ, $\overline{\Delta}^2$, ζ or η is in general stronger on R than on T. To be more quantitative (see tables 1, 2, 3), oscillation effects are up to 50% for pion sources and almost a factor of two for muon-damped and neutron sources. Impurities in the initial flux composition of 0.1 lead to deviations in the flux ratios of 4%, 20% and 15% for pion, muon-damped and neutron sources, respectively.

We gave several illustrative examples where the assumption of an idealized, pure initial flux ratio may easily (mis)lead to incorrect inferences. Wrong inferences may include the octant of θ_{23}, the magnitude of $|U_{e3}|$ and the existence of leptonic CP-violation. We stress that in future analyses, the intrinsic flux uncertainty should be taken into account before inferences are drawn.

Acknowledgments

A. Mixing probabilities

For the sake of completeness, we give here the explicit forms of the oscillation probabilities. The lepton mixing, or PMNS, matrix is parameterized in Particle Data Group format as

$$
U = \begin{pmatrix} c_{12}\,c_{13} & s_{12}\,c_{13} & s_{13}\,e^{-i\delta} \\ -s_{12}\,c_{23} - c_{12}\,s_{23}\,s_{13}\,e^{i\delta} & c_{12}\,c_{23} - s_{12}\,s_{23}\,s_{13}\,e^{i\delta} & s_{23}\,c_{13} \\ s_{12}\,s_{23} - c_{12}\,c_{23}\,s_{13}\,e^{i\delta} & -c_{12}\,s_{23} - s_{12}\,c_{23}\,s_{13}\,e^{i\delta} & c_{23}\,c_{13} \end{pmatrix}\,, \qquad (A.1)
$$

where $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$, and we have omitted the Majorana phases, which are irrelevant for neutrino oscillations. Using eq. (2.1), one obtains

$$P_{e\mu} = 2\,c_{13}^2\,\big\{c_{12}^2\,s_{12}^2\,c_{23}^2 + \big(c_{12}^4 + s_{12}^2\big)\,s_{13}^2\,s_{23}^2 \tag{A.2}$$
$$+\,c_{12}\,s_{12}\,c_{23}\,s_{23}\,c_\delta\,(c_{12}^2 - s_{12}^2)\,s_{13}\big\}\,,$$

where $c_\delta = \cos\delta$, and

$$P_{\mu\mu} = 1 - 2\,c_{12}^4\,c_{23}^2\,s_{23}^2\,s_{13}^2 \tag{A.3}$$
$$+2\,\big\{\big(s_{12}^2\,\big[\big(s_{13}^4 + [4\,c_\delta^2 - 1]\,s_{13}^2 + 1\big)\,s_{23}^2 - 1\big] - c_{13}^2\,s_{23}^2\big)\,c_{23}^2$$
$$+s_{13}^2\,s_{23}^2\,(c_{13}^2\,s_{12}^2 - [c_{13}^2 + s_{12}^2]\,s_{23}^2)\big]\,c_{12}^2$$
$$+s_{23}\,\big[-2\,\big(c_{23}^2\,c_{13}^4 + [c_{13}^2 + c_{23}^2\,s_{12}^2]\,s_{13}^2\big)\,s_{23}\,s_{12}^2$$
$$-2\,c_{12}\,s_{12}\,c_{23}\,c_\delta\,(c_{12}^2 - s_{12}^2)\,(c_{13}^2 + [s_{13}^2 + 1]\,[c_{23}^2 - s_{23}^2])\,s_{13}\big\}\,.$$

With the help of the identities [45, 11]

$$P_{e\tau} = P_{e\mu}(\theta_{23} \to \theta_{23} + \pi/2 \text{ or } \theta_{23} \to \theta_{23} + 3\pi/2)\,, \tag{A.4}$$
$$P_{\tau\tau} = P_{\mu\mu}(\theta_{23} \to \theta_{23} + \pi/2 \text{ or } \theta_{23} \to \theta_{23} + 3\pi/2)\,,$$

and the unitary relations

$$P_{ee} = 1 - P_{e\mu} - P_{e\tau}\,, \tag{A.5}$$
$$P_{\mu\tau} = 1 - P_{e\mu} - P_{\mu\mu}\,, \tag{A.6}$$
$$P_{\tau\tau} = 1 - P_{e\tau} - P_{\mu\tau} = P_{ee} + 2\,P_{e\mu} + P_{\mu\mu} - 1 = P_{e\mu} - P_{e\tau} + P_{\mu\mu}\,, \tag{A.7}$$

all other probabilities can be readily obtained.

References

[1] J.G. Learned and S. Pakvasa, *Detecting tau-neutrino oscillations at PeV energies*, *Astropart. Phys.* **3** (1995) 267 [hep-ph/9405296].

[2] S. Pakvasa, *Neutrino properties from high energy astrophysical neutrinos*, *Mod. Phys. Lett.* **A 19** (2004) 1163 [*Yad. Fiz.* **67** (2004) 1179] [hep-ph/0405179].

[3] P. Bhattacharjee and N. Gupta, *Probing neutrino mixing angles with ultrahigh energy neutrino telescopes*, hep-ph/0501191.

[4] P.D. Serpico and M. Kachelriess, *Measuring the 13-mixing angle and the CP phase with neutrino telescopes*, *Phys. Rev. Lett.* **94** (2005) 211102 [hep-ph/0502088].

[5] P.D. Serpico, *Probing the 2-3 leptonic mixing at high-energy neutrino telescopes*, *Phys. Rev.* **D 73** (2006) 047301 [hep-ph/0511313].

[6] Z.-Z. Xing and S. Zhou, *Towards determination of the initial flavor composition of ultrahigh-energy neutrino fluxes with neutrino telescopes*, *Phys. Rev.* **D 74** (2006) 013010 [astro-ph/0603781].

[7] W. Winter, *How astrophysical neutrino sources could be used for early measurements of neutrino mass hierarchy and leptonic CP phase*, *Phys. Rev.* **D 74** (2006) 033015 [hep-ph/0604191].

[8] Z.-z. Xing, *Neutrino telescopes as a probe of broken μ τ symmetry*, *Phys. Rev.* **D 74** (2006) 013009 [`hep-ph/0605219`].

[9] M. Kachelriess and R. Tomas, *High energy neutrino yields from astrophysical sources. I: weakly magnetized sources*, *Phys. Rev.* **D 74** (2006) 063009 [`astro-ph/0606406`].

[10] D. Majumdar and A. Ghosal, *Probing deviations from TriBiMaximal mixing through ultra high energy neutrino signals*, *Phys. Rev.* **D 75** (2007) 113004 [`hep-ph/0608334`].

[11] W. Rodejohann, *Neutrino mixing and neutrino telescopes*, *JCAP* **01** (2007) 029 [`hep-ph/0612047`].

[12] D. Meloni and T. Ohlsson, *Neutrino flux ratios at neutrino telescopes: the role of uncertainties of neutrino mixing parameters and applications to neutrino decay*, *Phys. Rev.* **D 75** (2007) 125017 [`hep-ph/0612279`].

[13] K. Blum, Y. Nir and E. Waxman, *Probing CP-violation in neutrino oscillations with neutrino telescopes*, `arXiv:0706.2070`.

[14] M. Kachelriess, S. Ostapchenko and R. Tomas, *High energy neutrino yields from astrophysical sources II: magnetized sources*, `arXiv:0708.3047`.

[15] G.-R. Hwang and K. Siyeon, *Neutrino telescopes and the degeneracy problem*, `arXiv:0711.3122`.

[16] THE ICECUBE collaboration, J. Ahrens et al., *IceCube: the next generation neutrino telescope at the south pole*, *Nucl. Phys.* **118** *(Proc. Suppl.)* (2003) 388 [`astro-ph/0209556`].

[17] Information available at http://www.km3net.org.

[18] J.F. Beacom, N.F. Bell, D. Hooper, S. Pakvasa and T.J. Weiler, *Measuring flavor ratios of high-energy astrophysical neutrinos*, *Phys. Rev.* **D 68** (2003) 093005 [*Erratum ibid.* **D 72** (2005) 019901] [`hep-ph/0307025`].

[19] J.F. Beacom, N.F. Bell, D. Hooper, S. Pakvasa and T.J. Weiler, *Decay of high-energy astrophysical neutrinos*, *Phys. Rev. Lett.* **90** (2003) 181301 [`hep-ph/0211305`]; *Sensitivity to θ_{13} and δ in the decaying astrophysical neutrino scenario*, *Phys. Rev.* **D 69** (2004) 017303 [`hep-ph/0309267`].

[20] J.F. Beacom et al., *Pseudo-Dirac neutrinos, a challenge for neutrino telescopes*, *Phys. Rev. Lett.* **92** (2004) 011101 [`hep-ph/0307151`];
P. Keranen, J. Maalampi, M. Myyrylainen and J. Riittinen, *Effects of sterile neutrinos on the ultrahigh-energy cosmic neutrino flux*, *Phys. Lett.* **B 574** (2003) 162 [`hep-ph/0307041`].

[21] K. Enqvist, P. Keranen and J. Maalampi, *Testing neutrino magnetic moments with AGNs*, *Phys. Lett.* **B 438** (1998) 295 [`hep-ph/9806392`];
G. Domokos and S. Kovesi-Domokos, *Neutrino flavor conversion in random magnetic fields*, *Phys. Lett.* **B 410** (1997) 57 [`hep-ph/9703265`].

[22] P.Q. Hung and H. Pas, *Cosmo MSW effect for mass varying neutrinos*, *Mod. Phys. Lett.* **A 20** (2005) 1209 [`astro-ph/0311131`].

[23] D. Hooper, D. Morgan and E. Winstanley, *Probing quantum decoherence with high-energy neutrinos*, *Phys. Lett.* **B 609** (2005) 206 [`hep-ph/0410094`]; *Lorentz and CPT invariance violation in high-energy neutrinos*, *Phys. Rev.* **D 72** (2005) 065009 [`hep-ph/0506091`];

M.C. Gonzalez-Garcia, F. Halzen and M. Maltoni, *Physics reach of high-energy and high-statistics IceCube atmospheric neutrino data*, *Phys. Rev.* **D 71** (2005) 093010 [hep-ph/0502223];
L.A. Anchordoqui, H. Goldberg, M.C. Gonzalez-Garcia, F. Halzen, D. Hooper, S. Sarkar and T. J. Weiler, *Probing Planck scale physics with IceCube*, *Phys. Rev.* **D 72** (2005) 065019 [hep-ph/0506168].

[24] H. Minakata and A.Y. Smirnov, *High energy cosmic neutrinos and the equivalence principle*, *Phys. Rev.* **D 54** (1996) 3698 [hep-ph/9601311].

[25] M.C. Gonzalez-Garcia and M. Maltoni, *Phenomenology with massive neutrinos*, arXiv:0704.1800.

[26] R.N. Mohapatra et al., *Theory of neutrinos: a white paper*, *Rept. Prog. Phys.* **70** (2007) 1757 [hep-ph/0510213];
R.N. Mohapatra and A.Y. Smirnov, *Neutrino mass and new physics*, *Ann. Rev. Nucl. Part. Sci.* **56** (2006) 569 [hep-ph/0603118];
A. Strumia and F. Vissani, *Neutrino masses and mixings and...*, hep-ph/0606054.

[27] P.F. Harrison, D.H. Perkins and W.G. Scott, *TriBiMaximal mixing and the neutrino oscillation data*, *Phys. Lett.* **B 530** (2002) 167 [hep-ph/0202074];
P.F. Harrison and W.G. Scott, *Symmetries and generalizations of TriBiMaximal neutrino mixing*, *Phys. Lett.* **B 535** (2002) 163 [hep-ph/0203209];
Z.Z. Xing, *Nearly TriBiMaximal neutrino mixing and CP violation*, *Phys. Lett.* **B 533** (2002) 85 [hep-ph/0204049];
X.G. He and A. Zee, *Some simple mixing and mass matrices for neutrinos*, *Phys. Lett.* **B 560** (2003) 87 [hep-ph/0301092];
L. Wolfenstein, *Oscillations among three neutrino types and CP violation*, *Phys. Rev.* **D 18** (1978) 958;
Y. Yamanaka, H. Sugawara and S. Pakvasa, *Permutation symmetries and the fermion mass matrix*, *Phys. Rev.* **D 25** (1982) 1895 [*Erratum ibid.* **D 29** (1984) 2135].

[28] L.A. Anchordoqui, H. Goldberg, F. Halzen and T.J. Weiler, *Neutrinos as a diagnostic of high energy astrophysical processes*, *Phys. Lett.* **B 621** (2005) 18 [hep-ph/0410003].

[29] A.P. Szabo and R.J. Protheroe, *Implications of particle acceleration in active galactic nuclei for cosmic rays and high-energy neutrino astronomy*, *Astropart. Phys.* **2** (1994) 375 [astro-ph/9405020].

[30] J.P. Rachen and P. Meszaros, *Photohadronic neutrinos from transients in astrophysical sources*, *Phys. Rev.* **D 58** (1998) 123005 [astro-ph/9802280].

[31] A. Muecke and R.J. Protheroe, *Modeling the april* 1997 *flare of MKN* 501, astro-ph/9910460; *A proton synchrotron blazar model for flaring in Markarian* 501, *Astropart. Phys.* **15** (2001) 121 [astro-ph/0004052].

[32] A. Muecke, R.J. Protheroe, R. Engel, J.P. Rachen and T. Stanev, *BL Lac objects in the synchrotron proton blazar model*, *Astropart. Phys.* **18** (2003) 593 [astro-ph/0206164].

[33] T. Kashti and E. Waxman, *Flavoring astrophysical neutrinos: flavor ratios depend on energy*, *Phys. Rev. Lett.* **95** (2005) 181101 [astro-ph/0507599].

[34] P. Lipari, M. Lusignoli and D. Meloni, *Flavor composition and energy spectrum of astrophysical neutrinos*, *Phys. Rev.* **D 75** (2007) 123005 [arXiv:0704.0718].

[35] L.A. Anchordoqui, H. Goldberg, F. Halzen and T.J. Weiler, *Galactic point sources of TeV antineutrinos*, *Phys. Lett.* **B 593** (2004) 42 [astro-ph/0311002].

[36] R. Engel, D. Seckel and T. Stanev, *Neutrinos from propagation of ultra-high energy protons*, *Phys. Rev.* **D 64** (2001) 093010 [astro-ph/0101216].

[37] S. Pakvasa, W. Rodejohann and T.J. Weiler, *Triminimal parametrization of the neutrino mixing matrix*, arXiv:0711.0052.

[38] P.F. Harrison and W.G. Scott, *μ-τ reflection symmetry in lepton mixing and neutrino oscillations*, *Phys. Lett.* **B 547** (2002) 219 [hep-ph/0210197].

[39] L. Anchordoqui and F. Halzen, *IceHEP high energy physics at the south pole*, *Ann. Phys. (NY)* **321** (2006) 2660 [hep-ph/0510389].

[40] S. Razzaque, P. Meszaros and E. Waxman, *Enhancement of the $\bar{\nu}/e$ flux from astrophysical sources by two photon annihilation interactions*, *Phys. Rev.* **D 73** (2006) 103005 [astro-ph/0509186].

[41] M. Sorel, J.M. Conrad and M. Shaevitz, *A combined analysis of short-baseline neutrino experiments in the (3 + 1) and (3 + 2) sterile neutrino oscillation hypotheses*, *Phys. Rev.* **D 70** (2004) 073004 [hep-ph/0305255];
G. Karagiorgi et al., *Leptonic CP-violation studies at MiniBooNE in the (3 + 2) sterile neutrino oscillation hypothesis*, *Phys. Rev.* **D 75** (2007) 013011 [hep-ph/0609177];
M. Maltoni and T. Schwetz, *Sterile neutrino oscillations after first MiniBooNE results*, *Phys. Rev.* **D 76** (2007) 093005 [arXiv:0705.0107].

[42] S. Goswami and W. Rodejohann, *Miniboone results and neutrino schemes with 2 sterile neutrinos: possible mass orderings and observables related to neutrino masses*, *JHEP* **10** (2007) 073 [arXiv:0706.1462].

[43] The MiniBooNE collaboration, A.A. Aguilar-Arevalo et al., *A search for electron neutrino appearance at the $\delta m^2 \sim 1 ev^2$ scale*, *Phys. Rev. Lett.* **98** (2007) 231801 [arXiv:0704.1500].

[44] H. Athar, M. Jezabek and O. Yasuda, *Effects of neutrino mixing on high-energy cosmic neutrino flux*, *Phys. Rev.* **D 62** (2000) 103007 [hep-ph/0005104];
R.L. Awasthi and S. Choubey, *Confusing sterile neutrinos with deviation from TriBiMaximal mixing at neutrino telescopes*, arXiv:0706.0399.

[45] E.K. Akhmedov, R. Johansson, M. Lindner, T. Ohlsson and T. Schwetz, *Series expansions for three-flavor neutrino oscillation probabilities in matter*, *JHEP* **04** (2004) 078 [hep-ph/0402175].